A model for competitive markets

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MATH 345 Differential Equations

Contents

In these notes we will present a simple, but remarkable model for the dynamics of prices under the law of supply and demand. We will show that under reasonable hypotheses for a competitive market, if there exists an equilibrium scenario, then this scenario is stable and asymptotically stable (in the large time limit). This is a well-known result due to Arrow, Block and Hurwicz. These notes are mostly based on M. Girotti's personal notes from the course "Fisica Matematica I" given by Prof. Dario Bamusi at Università degli Studi di Milano in far 2007 and on the seminal paper [1].

1 Competitive markets

Let's consider a market where there is a certain number of individuals who trades (i.e. buys and sells) N goods among themselves. Each good will have a certain price which can vary in time: $p_i = p_i(t)$, for $i = 1, \dots, N$.

Hypothesis 1 (Competitivity): we are assuming we are studying a **capitalistic market**, where nothing is given for free. Therefore, the prices will never be equal to zero:

$$p_i(t) > 0 \qquad \forall\ i = 1, \dots, N,\ \forall\ t \in R.$$

We can collect the prices of all the goods under consideration in a **price vector**

$$\mathbf{p} = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_N \end{bmatrix}.$$

Each person in the market is both buying and selling all the goods; in particular, the j-th person will have a demand of the j-th good equal to some value d_{ij} and he/she will have a supply of the same good equal to another value s_{ij}:

$$d_{ij} = d_{ij}(\mathbf{p}) \quad \rightarrow \text{demand of the } i\text{-th good by the } j\text{-th person}$$
$$s_{ij} = s_{ij}(\mathbf{p}) \quad \rightarrow \text{supply of the } i\text{-th good by the } j\text{-th person}$$

All these values are clearly dependent on the prices of the goods themselves (how expensive or how cheap they are).

We can collect all these values in a "demand table" as well as in a "supply table" as shown in Figure 1. The total demand of the i-th good will them be the sum over i-th row of the "demand table":

$$d_i(\mathbf{p}) = \sum_{j=1}^{m} d_{ij}$$

similarly the total supply of the i-th good will be the sum over the i-th row of the "supply table":

$$s_i(\mathbf{p}) = \sum_{j=1}^{m} s_{ij}$$

	Person 1	Person 2	...	Person m
good #1	d_{11}	d_{12}	...	d_{1m}
good #2	d_{21}	d_{22}	...	d_{2m}
$\vdots$	...			
good #N	d_{N1}	d_{N2}	...	d_{Nm}

Figure 1: An example of a Demand Table with m people and N goods. The total demand of each good is equal to the sum over the corresponding row.

Our capitalistic market will clearly be based on the **Law of Supply and Demand**: if there is more demand of one item than its supply, its price will then increase. Vice versa, if there is more supply than demand, the price of the good will decrease.

Let

$$F_i(\mathbf{p}) = d_i(\mathbf{p}) - s_i(\mathbf{p})$$

a function that keeps track of the surplus of demand for the good number i. Then, according to the Law of Supply and Demand, the variation of price of that good will be dictated by how much excess (or shortage) of demand we have at any time:

$$p_i' = F_i(\mathbf{p}).$$

It is easy to notice that if there is an excess in demand and not enough supply, then $F_i(\mathbf{p}) > 0$ and the price of the i-th good grows; on the other hand, if there is a lot of supply and not much demand, then $F_i(\mathbf{p}) < 0$ and the price the good decreases.

1.1 Competitive market with only one good

Let's consider this toy model where there are a certain number of people trading only one good (say, strawberries) at price $p = p(t)$. The price equation is

$$p' = F(p) = d(p) - s(p)$$

First of all, we can notice that the equation is a separable ODE, therefore a solution can easily be found, once the functions $d(p)$ and $s(p)$ are explicit:

$$\int_{p(t_0)}^{p(t)} \frac{\mathrm{d}p}{d(p) - s(p)} = \int \mathrm{d}t = t - t_0$$

On the other hand, regardless of whether we know how $d(p)$ and $s(p)$ look like, we can still qualitatively analyze this equation.

- The function $y = d(p)$ is called **demand curve**. Under the law of Supply and Demand it is reasonable to assume that the function has the following properties:
  - $d(p) \geq 0$ for all prices $p \in \mathbb{R}_+$; at most, the demand could be equal to zero, but it cannot be negative;
  - $d(0) > 0$, the demand for some free goods is always high (!);
  - the demand $d(p)$ is decreasing as the price p is increasing.
- The function $y = s(p)$ is called **supply curve**. We can also assume similar properties for such a function:
  - $s(p) \geq 0$ for all prices $p \in \mathbb{R}_+$; at most, the supplly could be equal to zero;
  - $s(0) = 0$, we assumed no giveaways (nobody wants to give goods for free);
  - the supply $s(p)$ is increasing as the price p is increasing (because there would be less people willing to buy).

Collecting all these information, we can give a rough sketch of how these functions could look like as in Figure 2.

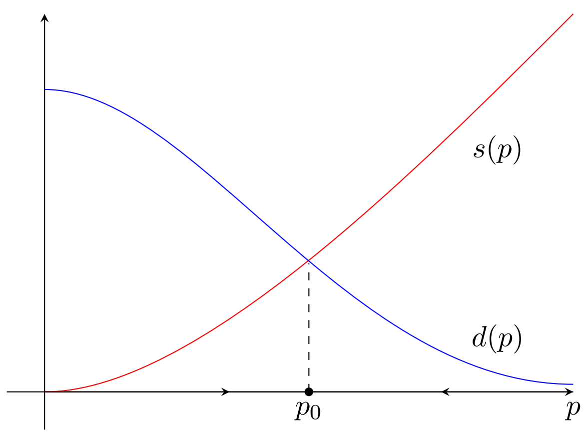

Figure 2: The demand and supply curves. It is clear from the behaviour of the functions that p_0 is a attractive equilibrium point.

Since $d(p)$ is a decreasing function ond $s(p)$ is increasing, there exists at least one point p_0 where the two curves cross each other:

$$d(p_0) = s(p_0), \quad \text{meaning} \quad F(p_0) = d(p_0) - s(p_0) = 0.$$

This means that $p(t) = p_0$ is an equilibrium point(equilibrium solution) for the differential equation $p' = F(p)$.

Let's analyze the stability of p_0: for $p < p_0$ we have that $F(p) = d(p) - s(p) > 0$, by simply looking at the graph, and for $p > p_0$ $F(p) < 0$. This means that for the values of p before p_0 the dynamics is moving forward and then $p > p_0$ the dynamics is moving backwards (see Figure 2): p_0 is a stable and asymptotically stable equilibrium point.

1.2 Competitive market with N goods

For a more general scenario, we need to introduce some extra hypotheses and constraints that will still be derived using common sense, while trying to keep the model relatively simple.

Since we are considering N goods, each good's price will have its own differential equation to satisfy, but the function F which measures the excess of demand will now depend on the prices of all the goods: $F_i = F_i(p_1, p_2, \ldots, p_N)$ for each good $i = 1, \ldots, N$. Therefore we have a system of N equation for the unknown functions $p_1(t), \ldots, p_N(t)$. We can write it in a compact form as

$$\mathbf{p}' = \mathbf{F}(\mathbf{p}).$$

Hypothesis 2 (Budget Constraints). We are assuming that nobody can cheat, claiming he/she has more money than what he/she actually has, and nobody loans money to other people. In this setting, each individual j has some amount of items of each good $s_{ij}(\mathbf{p})$ that he/she can sell at price p_i, for each good $i = 1, \ldots, N$.

The total (virtual) amount of money that the individual j can gain is then

$$b_j = \sum_{i=1}^{N} s_{ij}(\mathbf{p}) p_i \qquad \text{the total budget.}$$

Hypothesis 3 (Maximal Satisfaction). We are further assuming that each person spends all the budget that he/she has immediately! (we are not allowing savings)

This means that the total expenses of the j person

$$e_j = \sum_{i=1}^{N} d_{ij}(\mathbf{p}) p_i$$

will be exactly equal to the total budget: $e_j = b_j$.

This implies that for each j

$$0 = e_j - b_j = \sum_{i=1}^{N} d_{ij}(\mathbf{p}) p_i - \sum_{i=1}^{N} s_{ij}(\mathbf{p}) p_i = \sum_{i=1}^{N} \left[d_{ij}(\mathbf{p}) - s_{ij}(\mathbf{p}) \right] p_i = \sum_{i=1}^{N} F_i(\mathbf{p}) p_i.$$

This relation is known under the name of **Walras Law**:

$$\mathbf{p} \cdot \mathbf{F}(\mathbf{p}) = 0$$

Hypothesis 4 (Homogeneity). It is reasonable to assume changing the currency used in the trading doesn't affect the behaviour of the market (as a first approximation of the model). In mathematics terms, we require the function $\mathbf{F}(\mathbf{p})$ to be homogeneous:

$$\mathbf{F}(\lambda\mathbf{p}) = \mathbf{F}(\mathbf{p}) \qquad \forall\ \lambda \in \mathbb{R}_+.$$

Hypothesis 5 (Gross Substitutability). Roughly speaking, we are assuming that if all the prices are fixed except the price of one good, say the k-th good, then the demand of all the other goods different from the k-th one will increases:

$$\frac{\partial F_i}{\partial p_k}(\mathbf{p}) > 0 \qquad \forall i \neq k$$

You can imagine a case where you would like to buy strawberries, but their price increased since last time you did the groceries and now they're too expensive. So, instead, you decide to buy apples because their price is still the same as last time, and maybe a couple of onions. Then, the demand of both the apples and the onions increases!

The following results hold.

Proposition 1. *Given a competitive market*

$$\mathbf{p}' = \mathbf{F}(\mathbf{p}).$$

and under the Hypotheses 4 and 5 above, if there exists an equilibrium point π (i.e. $\mathbf{F}(\pi) = \mathbf{0}$), then it is unique (up to a scaling factor $\lambda\pi$, $\lambda \in \mathbb{R}_+$).

Proposition 2. *Given a competitive market*

$$\mathbf{p}' = \mathbf{F}(\mathbf{p}).$$

and under the assumption that Walras Law holds, then

$$\|\mathbf{p}\| = \sqrt{\sum_{i=1}^{N} p_i^2}$$

is a constant of motion.

We will not give the proof of Proposition 1. The proof of Proposition 2 on the other hand is very simple.

Proof. We simply need to show that the quantity $\|\mathbf{p}(t)\|$ does not depend on time. For simplicity, we can consider the norm squared and calculate its time derivative:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\mathbf{p}\|^2\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{i=1}^{N} p_i^2\right) = \sum_{i=1}^{N} 2p_i p_i' = 2\sum_{i=1}^{N} p_i F_i(\mathbf{p}) = 0$$

where the last equality is due to Walras Law. □

An important consequence of Proposition 2 is the following:

once we fix the initial condition $\mathbf{p}(0) = \mathbf{p}_0$, the price dynamics $\mathbf{p}(t)$ happens the (positive portion of the) sphere in $\mathbb{R}^N$ with radius $\|\mathbf{p}_0\|$, because the norm is a conserved quantity and it is the same for all times (see Figures 3 and 4).

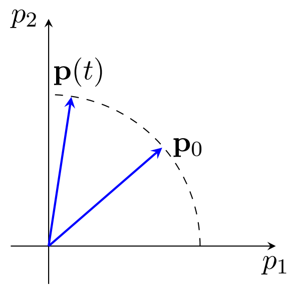

Figure 3: A market with $N = 2$ goods: the dynamics happens on a quarter of a circle.

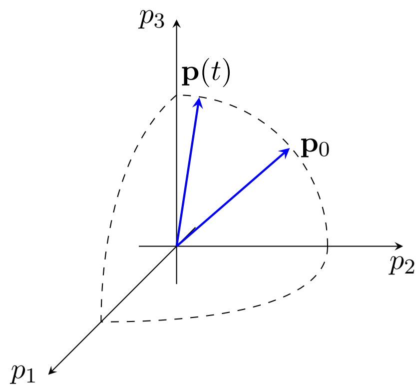

Figure 4: A market with $N = 3$ goods: the dynamics happens on the positive portion of a 3D-sphere.

2 Stability of the competitive market

Theorem 3 (**Arrow-Block-Hurwicz, 1959**). *Consider the price law*

$$\mathbf{p}' = \mathbf{F}(\mathbf{p})$$

and assume that the following hypotheses hold:

1. *Walras Law:* $\sum_{i=1}^{N} F_i(\mathbf{p}) p_i = 0$*;*

2. *Homogeneity:* $F_i(\lambda \mathbf{p}) = F_i(\mathbf{p})$ *for all* $i = 1, \ldots, N$*;*

3. *Gross Substitutability:* $\frac{\partial F_i}{\partial p_k}(\mathbf{p}) > 0$ *for all* $i \neq k$, $i, k = 1, \ldots, N$.

If there exists an equilibrium point

$$\pi \in \mathbb{R}^N_+ \quad \textit{such that } \mathbf{F}(\pi) = \mathbf{0}$$

then, π *is attractive (i.e. the constant solution* $\mathbf{p}(t) = \pi \ \forall \ t \in \mathbb{R}_+$ *is asymptotically stable).*

A striking consequence of this theorem is the following:

in a competitive market, if there exists some equilibrium value for the prices of the goods we are considering, then the prices will converge to that equilibrium value in the long run, implying that there will be a stable balance of supply and demand.

The proof is very simple and it just requires the use of some intuitive tools in linear algebra and basic calculus.

Proof. Let $\mathbf{p}_0 \in \mathbb{R}^N$ be the initial condition of our Cauchy problem:

$$\begin{cases} \mathbf{p}' = \mathbf{F}(\mathbf{p}) \\ \mathbf{p}(0) = \mathbf{p}_0 \end{cases}$$

and let $\pi \in \mathbb{R}^N$ be an equilibrium point of the system (i.e. $\mathbf{F}(\pi) = \mathbf{0}$). We can assume, thanks to homogeneity, that $\|\pi\| = \|\mathbf{p}_0\|$.

We consider now the solution $\mathbf{p}(t)$ of the above Cauchy problem with initial condition $\mathbf{p}(0) = \mathbf{p}_0$ and we want to study the distance between the solution $\mathbf{p}(t)$ and the equilibrium point π:

$$\mathrm{dist}^2\left(\mathbf{p}(t), \pi\right) = \|\mathbf{p}(t) - \pi\|^2 \geq 0.$$

Our goal is to prove that as $t \to +\infty$, this quantity tends to zero, implying that the solution $\mathbf{p}(t)$ converges to the equilibrium solution π.

We then study its derivative:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(\|\mathbf{p}(t) - \pi\|^2\right) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{i=1}^N (p_i(t) - \pi_i)^2\right) = 2\sum_{i=1}^N (p_i(t) - \pi_i)\, p_i' = 2\sum_{i=1}^N (p_i(t) - \pi_i)\, F_i(\mathbf{p}) \\ &= 2\sum_{i=1}^N p_i F_i(\mathbf{p}) - 2\sum_{i=1}^N \pi_i F_i(\mathbf{p}) = -2\pi\sum_{i=1}^N F_i(\mathbf{p})\pi_i \end{aligned}$$

where in the last equality we used Walras Law.

It is clear that if we manage to prove that if $\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\mathbf{p}(t) - \pi\|^2\right) < 0$, then the function $\|\mathbf{p}(t) - \pi\|^2$ decreases with time and, since it is a non-negative function by construction, it necessarily follows that $\|\mathbf{p}(t) - \pi\|^2 \to 0$ as $t \to +\infty$.

To get what we want, we will use the following

Lemma 4. *In the same hypotheses of the theorem above, we have*

$$\sum_{i=1}^N F_i(\mathbf{p})\pi_i > 0$$

Proof. We will only give a proof in the case of $N = 2$ goods. The proof for general N is more complicated to visualize, but it follows the same idea.

Set

$$\pi = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \qquad \text{and} \qquad \mathbf{p} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$$

where $\alpha, \beta, p_1, p_2 \geq 0$ and $\mathbf{p}$ is another solution, different from π and (by homogeneity) also not proportional to π: $\mathbf{p} \neq \lambda\pi$ for any $\lambda \in \mathbb{R} \setminus 0$. However, we can still assume that both vectors have the same norm (the same length): $\|\mathbf{p}\| = \|\pi\|$.

This implies that

$$\frac{p_1}{p_2} \neq \frac{\alpha}{\beta}$$

meaning that the slope of the two vectors are different.

Suppose that

$$\frac{p_1}{p_2} < \frac{\alpha}{\beta}$$

(the case where $\frac{p_1}{p_2} > \frac{\alpha}{\beta}$ follows the same argument), meaning that the slope of $\mathbf{p}$ is bigger than the slope of π, as in Figure 5.

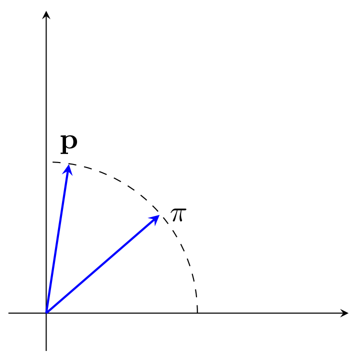

Figure 5: The case $\frac{p_1}{p_2} < \frac{\alpha}{\beta}$.

By Walras Law,

$$\mathbf{p} \cdot \mathbf{F}(\mathbf{p}) = p_1 F_1(\mathbf{p}) + p_2 F_2(\mathbf{p}) = 0$$

meaning that the vector $\mathbf{p}$ and the vector $\mathbf{F}(\mathbf{p})$ are orthogonal. Moreover, the vector $\mathbf{F}(\mathbf{p})$ lies in the fourth quadrant: indeed, consider the vector $\mathbf{v} = \mu\mathbf{p}$ with $\mu = \frac{\beta}{p_2}$

$$\mathbf{v} = \begin{bmatrix} \mu p_1 \\ \mu p_2 \end{bmatrix} = \begin{bmatrix} \frac{\beta}{p_2} p_1 \\ \beta \end{bmatrix} < \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \pi$$

where the inequality is given by the fact that the second component is unchanged and for the first component we used the hypothesis $\frac{p_1}{p_2} < \frac{\alpha}{\beta}$.

By Gross Substitutability, we have that

$$\frac{\partial F_2}{\partial p_1} > 0$$

meaning that $F_2(\mathbf{p})$ increases if the first component of $\mathbf{p}$ increases. Therefore,

$$F_2(\mathbf{p}) = F_2(\mu\mathbf{p}) = F_2\left(\begin{bmatrix} \frac{\beta}{p_2} p_1 \\ \beta \end{bmatrix}\right) < F_2\left(\begin{bmatrix} \alpha \\ \beta \end{bmatrix}\right) = F_2(\pi) = 0$$

where we used Homogeneity in the first equality and the fact that π is an equilibrium point in the last equality. This shows that the second component of the vector $\mathbf{F}(\mathbf{p})$ is negative, therefore the vector itself must be in the fourth quadrant (see Figure 6).

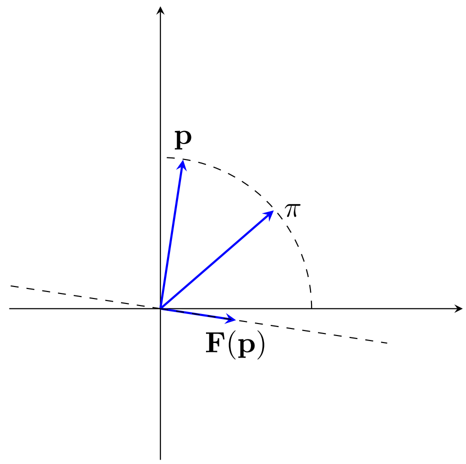

Figure 6: The location of the vector $\mathbf{F}(\mathbf{p})$ in the case $\frac{p_1}{p_2} < \frac{\alpha}{\beta}$.

Now, collecting all these observation, we have:

$$\sum_{i=1}^{2} F_i(\mathbf{p})\pi_i = \pi \cdot \mathbf{F}(\mathbf{p}) = \|\pi\| \, \|\mathbf{F}(\mathbf{p})\| \cos(\theta) > 0$$

because θ is the angle between the two vectors and $\mathbf{F}(\mathbf{p})$ is in the fourth quadrant while π is in the first one (thus, $\theta \in [0, \frac{\pi}{2})$ is an acute angle). $\square$

Finally, using the Lemma we just proved, it is clear that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\mathbf{p}(t) - \pi\|^2\right) < 0.$$

Therefore, the function $\|\mathbf{p}(t) - \pi\|^2$ decreases with time and

$$\|\mathbf{p}(t) - \pi\|^2 \to 0 \qquad \text{as } t \to +\infty,$$

meaning that π is an asymptotically stable equilibrium solution:

$$\mathbf{p}(t) \to \pi \qquad \text{as } t \to +\infty.$$

$\square$

References

[1] K.J. Arrow, H.D. Block, and L. Hurwicz. On the stability of the competitive equilibrium, II. *Econometrica*, 27(1):82–109, 1959.