Addendum – Cantor's Theorem

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MATH 317-01 Advanced Calculus of one variable

Theorem 1 (Cantor)**.** $\mathbb{R}$ *is uncountable, i.e.* $\mathbb{R}$ **has the cardinality of continuum**: $|\mathbb{N}| < |\mathbb{R}|$.

Cantor's diagonalization process. We prove it by contradiction.

Assume that there exists a bijection $\mathbb{N} \leftrightarrow \mathbb{R}$. We then list all real numbers as

$$\begin{array}{ccr} 1 & \longleftrightarrow & 2.1972048... \\ 2 & \longleftrightarrow & 14.536613809... \\ 3 & \longleftrightarrow & 0.497310123... \\ 4 & \longleftrightarrow & 292.275818831... \\ 5 & \longleftrightarrow & 12.002200025... \\ 6 & \longleftrightarrow & 1.999902681... \\ \vdots & \longleftrightarrow & \vdots \end{array}$$

Now we construct a new rogue real number of the type

$$0.a_1a_2a_3a_4\cdots$$

that doesn't appear in the list and therefore cannot be put in correspondence with any natural number (we already used them all in the list above).

For the first digit a_1, we pick a digit that is different from the first decimal digit of the first number: in the example above, we want $a_1 \neq 1$ and we can choose $a_1 = 3$ for example.

For the second digit a_2 as well, we pick a digit that is different from the second decimal digit of the second number: in the example, we want $a_2 \neq 3$ and we can choose $a_2 = 9$ for example.

We continue this way for all the (infinite) digits of this new number that we want to build:

$$\begin{array}{ccr} 1 & \longleftrightarrow & 2.\mathbf{1}972048... \\ 2 & \longleftrightarrow & 14.5\mathbf{3}6613809... \\ 3 & \longleftrightarrow & 0.49\mathbf{7}310123... \\ 4 & \longleftrightarrow & 292.275\mathbf{8}18831... \\ 5 & \longleftrightarrow & 12.0022\mathbf{0}0025... \\ 6 & \longleftrightarrow & 1.99990\mathbf{2}681... \\ \vdots & \longleftrightarrow & \vdots \end{array}$$

The (real) number that we find at the end is not in the list by construction, because it differs by at least one digit from all the other numbers that are already in the list, and we cannot associate to it a natural number because they were all listed in the list above.

Therefore, there cannot exist a bijection between $\mathbb{N}$ and $\mathbb{R}$, but only a strict injection, meaning that

$$|\mathbb{N}| < |\mathbb{R}| .$$

□

Proposition 2. *Any open interval* $(a, b) \subset \mathbb{R}$ *(including the cases where* $a = -\infty$ *or* $b = +\infty$*) has the continuity of continuum:*

$$|(a, b)| = |\mathbb{R}| .$$

Sketch of the proof. There is a bijection between the interval $(0, 1)$ and the set $\mathbb{R}$ via the map

$$\begin{aligned} f : (0, 1) &\to \mathbb{R} \\ x &\mapsto \tan\left[\pi\left(x - \frac{1}{2}\right)\right] . \end{aligned}$$

Any other open interval (a, b) can be mapped into $(0, 1)$ via a linear transformation.

For unbounded intervals, the bijection is

$$\begin{aligned} f : (0, +\infty) &\to \mathbb{R} \\ x &\mapsto \log x \end{aligned}$$

and any other unbounded interval can be mapped into $(0, +\infty)$ via a linear transformation. □

Proposition 3. *Any (semi-)closed interval* $[a, b]$, $[a, b)$ *or* $(a, b]$ *(including the cases where* $a = -\infty$ *or* $b = +\infty$*) has the continuity of continuum:*

$$|[a, b]| = |\mathbb{R}| , \qquad |[a, b)| = |\mathbb{R}| , \qquad |(a, b]| = |\mathbb{R}| .$$