Addendum – Construction of $\mathbb{Q}$

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MATH 317-01 Advanced Calculus of one variable

1 Equivalence relations

Let X be a set.

Definition 1. An **equivalence relation** on the set X is a binary relation $\sim$ such that it is

1. reflexive: $\forall\, x \in X$, $x \sim x$
2. symmetric: $\forall\, x, y \in X$, if $x \sim y$, then $y \sim x$
3. transitive: $\forall\, x, y, z \in X$, if $x \sim y$ and $y \sim z$, then $x \sim z$.

Example 2. Let $X = \mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$ and define the relation

$$m \sim n \quad \text{for } m, n \in \mathbb{Z} \quad \Leftrightarrow \quad m - n \text{ is even}$$

Claim: $\sim$ is an equivalency relation.

Proof. First of all, we re-state the equivalence relation to have a clearer idea of what we need to prove: the equivalence relation $\sim$ is saying that two integers $m, n \in \mathbb{Z}$ are equivalent if and only if

$$m - n \text{ is divisible by 2, i.e. the difference can be written as } m - n = 2k \text{ for some } k \in \mathbb{Z}$$

We need to check the three properties listed above:

1. reflexive: $\forall\ n \in \mathbb{Z}$ we have that $n - n = 0 = 2 \cdot 0$, therefore $n \sim n$.
2. symmetric: $\forall\ m, n \in \mathbb{Z}$ such that $m \sim n$ (meaning $m - n = 2k$ for some $k \in \mathbb{Z}$), we have
$$n - m = -(m - n) = -(2k) = 2 \cdot (-k),$$
therefore $n \sim m$.
3. transitive: $\forall\ \ell, m, n \in \mathbb{Z}$ such that $\ell \sim m$ (meaning $\ell - m = 2k$ for some $k \in \mathbb{Z}$) and $m \sim n$ (meaning $m - n = 2h$ for some $h \in \mathbb{Z}$), we have
$$\ell - n = \ell - n + m - m = (\ell - m) + (m - n) = 2k + 2h = 2(k + h),$$
therefore $\ell \sim n$.

Definition 3. The set of elements in X that are all equivalent between each other is called **equivalence class** and it is denoted as

$$[x]_\sim := \left\{ y \in X \,\middle|\, x \sim y \right\}.$$

The collection of all the equivalence classes that can be found via an equivalence relation is called **quotient set** and it is denoted $X/\sim$.

Example 4 (cont'd). One equivalence class in the set $(\mathbb{Z}, \sim)$ is for example

$$[1]_\sim = \left\{ n \in \mathbb{Z} \,\middle|\, n \sim 1 \right\} = \left\{ n \in \mathbb{Z} \,\middle|\, n - 1 \text{ is even} \right\} = \left\{ n \in \mathbb{Z} \,\middle|\, n = \text{odd} \right\} = \left\{ 2k+1 \,\middle|\, k \in \mathbb{Z} \right\}$$

Another one (and there are no more) is

$$[0]_\sim = \left\{ n \in \mathbb{Z} \,\middle|\, n \sim 0 \right\} = \left\{ n \in \mathbb{Z} \,\middle|\, n - 0 \text{ is even} \right\} = \left\{ n \in \mathbb{Z} \,\middle|\, n = \text{even} \right\} = \left\{ 2k \,\middle|\, k \in \mathbb{Z} \right\}$$

The quotient set is $\mathbb{Z}/\sim = \{[0], [1]\}$. It is a well-known set and it is denoted as $\mathbb{Z}_2$.

An equivalence relation $\sim$ within a set X can be thought as sorting all the elements of the set X and grouping them into boxes according to a certain common characteristic (which is described by the equivalence relation itself). Each box is an equivalence class.

Clearly, all the elements that we put in one box (i.e. all the elements that are equivalent between each other, that share a common feature) are not equivalent to any other element from another box (they have different features).

Mathematically speaking, introducing an equivalence relation means to construct a partition of the set X (i.e. diving the set X into different boxes).

Definition 5. A **partition** $\mathcal{P}$ on a set X is a collection of non-empty subsets S of X such that

1. $X = \bigcup_{S \in \mathbb{P}} S$
2. if two subsets of the partition S_1 and S_2 have a common element ($S_1 \cap S_2 \neq \emptyset$), then they are the same subset $S_1 = S_2$.

Proposition 6. *The function $f : X \to X/\sim$, that maps $x \mapsto [x]_\sim$, is surjective.*

Remark 7. *The proposition claims that every element $x \in X$ belongs to one (and only one!) equivalence class. In particular, $x \in X$ cannot belong simultaneously to two different equivalence classes ("boxes").*

2 The rational numbers

Consider the set

$$X = \mathbb{Z} \times (\mathbb{Z} \backslash \{0\}) = \left\{ (a, b) \,\middle|\, a, b \in \mathbb{Z}, b \neq 0 \right\}$$

and introduce the equivalence relation

$$(a, b) \sim (c, d) \quad \Leftrightarrow \quad ad = bc$$

Claim: $\sim$ is an equivalence relation.

Proof: in this case there is no need to rewrite the equivalence relation, because it is already quite explicit.

We now check the three properties of an equivalence relation:

1. reflexive: $\forall\ (a,b) \in X$ we have that $ab = ba$, therefore $(a,b) \sim (a,b)$.
2. symmetric: $\forall\ (a,b),(c,d) \in X$ such that $(a,b) \sim (c,d)$ (meaning $ad = bc$), we have
$$cb = da,$$
therefore $(c,d) \sim (a,b)$.
3. transitive: $\forall\ (a,b),(c,d),(e,f) \in X$ such that $(a,b) \sim (c,d)$ (meaning $ad = bc$) and $(c,d) \sim (e,f)$ (meaning $cf = de$), we have
$$\begin{aligned} ad - bc &= 0 \\ cf - de &= 0; \end{aligned}$$
now multiply the first equation by f and the second one by b:
$$\begin{aligned} adf - bcf &= 0 \\ bcf - bde &= 0 \end{aligned}$$
and add them together:
$$adf - bcf + bcf - bde = adf - bde = d(af - be) = 0;$$
since $d \neq 0$ (d is the second element in the pair $(c,d) \in X$), we have
$$af - be = 0, \text{ i.e. } af = be,$$
therefore $(a,b) \sim (e,f)$.

Definition 8. The set of **rational numbers** $\mathbb{Q}$ is the quotient set $\mathbb{Z} \times (\mathbb{Z}\backslash\{0\})\,/\!\sim$. Each element in $\mathbb{Q}$ is an equivalence class of the form
$$[(a,b)]_\sim = \left\{ (c,d) \in \mathbb{Z} \times (\mathbb{Z}\backslash\{0\}) \;\middle|\; ad = bc \right\}$$
and it will denoted as
$$[(a,b)]_\sim =: \frac{a}{b}$$

Remark 9. *The set of integers $\mathbb{Z}$ is a subset of $\mathbb{Q}$ via the map $n \to [(n,1)]_\sim =: \frac{n}{1}$*

Important facts about $\mathbb{Q}$:

- $\mathbb{Q}$ can equipped with two operations called sum $+$ and product $\cdot$ (the usual sum and product) such that $(\mathbb{Q}, +, \cdot)$ is a **field** (Definition 1.1.5 from the book).

- $\mathbb{Q}$ has a natural ordering relation

$$\frac{p}{q} < \frac{m}{n}, \quad \text{for } \frac{p}{q}, \frac{m}{n} \in \mathbb{Q} \quad \Leftrightarrow \quad \frac{m}{n} - \frac{p}{q} > 0$$

  Moreover, this relation is **compatible with the field structure**, meaning that $(\mathbb{Q}, +, \cdot; <)$ is an **ordered field**. (Definition 1.1.7 from the book).

- $\mathbb{Q}$ is not complete, i.e. it does not satisfy the least-upper-bound property (Definition 1.1.3 from the book).

What now? The next step is to look for a "bigger" set that contains $\mathbb{Q}$ and has all the properties of $\mathbb{Q}$, i.e. an ordered field, with the additional property to be also complete.

Does it exist? Is there more than one set that we can define in this way? If there are more, what's the difference between them?