Addendum – Construction of $\mathbb{R}$ via Dedekind's method

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MATH 317-01 Advanced Calculus of one variable

The goal is to find a set F such that

1. $\mathbb{Q} \subset F$
2. F is an ordered set: $(F, <)$
3. F is equipped with a sum $+$ and a product $\cdot$ such that $(F, +\cdot)$ is a field and it is compatible with the ordering relation
4. F has the least upper bound property.

The simple idea due to R. Dedekind relies on the observation that every real number α is completely determined by all the rational numbers that are less than α and all the rational numbers that are greater or equal than α.

Step 0

We start with the obvious statement that $\forall\, r \in \mathbb{Q}$, r divides the set $\mathbb{Q}$ into two subsets

$$L_r := \left\{ q \in \mathbb{Q} \;\middle|\; q < r \right\}$$
$$U_r := \left\{ q \in \mathbb{Q} \;\middle|\; q \geq r \right\}$$

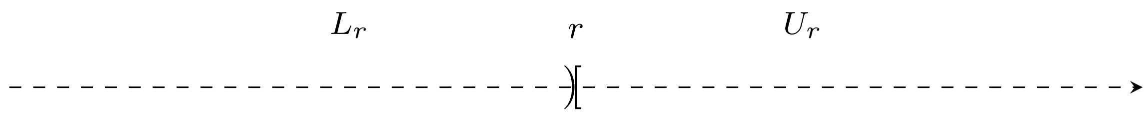
Definition 1. For each $r \in \mathbb{Q}$, we call **Dedekind cut** the pair (L_r, U_r).

Remark 2. *L_r doesn't have a maximum (rational) number, while U_r does have a minimum (rational) number and it is precisely the number r.*

Proposition 3. *The map*

$$\begin{aligned} \mathcal{D} : \mathbb{Q} &\to \{\text{Dedekind cuts}\} \\ r &\mapsto (L_r, U_r) \end{aligned}$$

is a bijection.



Step 1

Definition 4. A **(general) Dedekind cut** $\alpha = (L_\alpha, U_\alpha)$ is a subdivision of $\mathbb{Q}$ into two non-empty subsets L_α (lower interval) and U_α (upper interval) such that

(a) L_α has no maximum

(b) $\forall\, x \in L_\alpha$, $y \in U_\alpha$ we have $x < y$

Remark 5. *U_α could have minimum or not.*

In particular, if U_α has minimum, then α is a rational number, $\alpha \in \mathbb{Q}$ ((L_α, U_α) is a Dedekind cut as defined in Step 0). If U_α does not have a minimum, then we say that α is irrational.

Example 6. Consider $\alpha = (L_\alpha, U_\alpha)$ defined as

$$L_\alpha := \left\{ q \in \mathbb{Q} \;\middle|\; q < 0 \right\} \cup \left\{ q \in \mathbb{Q} \;\middle|\; q > 0, q^2 < 2 \right\}$$
$$U_\alpha := \left\{ q \in \mathbb{Q}_+ \;\middle|\; q^2 \geq 2 \right\}$$

Then, α is a (general) Dedekind cut, the minimum of U_α doesn't exist in $\mathbb{Q}$. A posteriori (once we'll be done with the construction of $\mathbb{R}$), we can say that α is actually the number $\sqrt{2}$.

Step 2

Definition 7. We define the set of **real numbers** $\mathbb{R}$ as

$$\mathbb{R} := \{\text{Dedekind cuts } \alpha = (L_\alpha, U_\alpha)\}$$

with the ordering relation

$$\alpha < \beta \text{ for any } \alpha, \beta \in \mathbb{R} \quad \Leftrightarrow \quad L_\alpha \subset L_\beta.$$

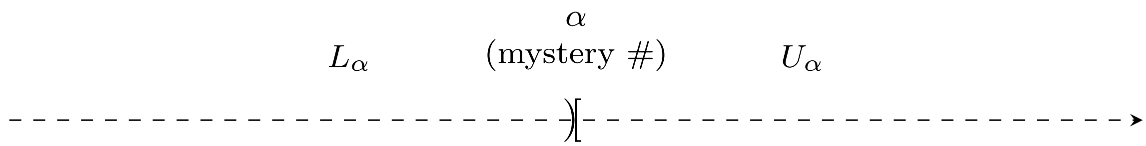
<u>Conclusion.</u> $(\mathbb{R}, <)$ is an ordered set that contains $\mathbb{Q}$. Indeed,

$$\mathbb{Q} \leftrightarrow \{\text{rational Dedekind cuts } r = (L_r, U_r)\} \hookrightarrow \{\text{Dedekind cuts } \alpha = (L_\alpha, U_\alpha)\} = \mathbb{R}$$

where $\leftrightarrow$ means that the map is a bijection and $\hookrightarrow$ means that the map is injective (it's an inclusion map).

Step 3

Define two operations $+$ (sum) and $\cdot$ (product).



Sum. $\forall\ \alpha, \beta \in \mathbb{R}$

$$\alpha + \beta := \left\{ x + y \;\middle|\; x \in L_\alpha, y \in L_\beta \right\}$$
$$0 := \left\{ q \in \mathbb{Q} \;\middle|\; q < 0 \right\}$$
$$-\alpha := \left\{ q \in \mathbb{Q} \;\middle|\; \exists\ r > 0 \text{ such that } -q - r \in U_\alpha \right\}$$

With these definitions we need to check that (*exercise*)

1. $+$ is commutative and associative
2. 0 is indeed the zero of the sum, i.e. $0 + \alpha = \alpha\ \forall\ \alpha \in \mathbb{R}$
3. $-\alpha$ is the inverse of α for the sum, i.e. $\alpha + (-\alpha) = 0\ \forall\ \alpha \in \mathbb{R}$.

Additionally, we need to check the compatibility with the ordering relation:

$$\forall\ \alpha, \beta, \gamma \in \mathbb{R} \text{ such that } \alpha < \beta, \text{ then } \alpha + \gamma < \beta + \gamma.$$

Product. First define the product for positive real numbers, i.e. $\forall\ \alpha, \beta \in \mathbb{R}_+ := \left\{ \alpha \in \mathbb{R} \;\middle|\; \alpha > 0 \right\}$ (where the 0 is the one defined above)

$$\alpha \cdot \beta := \left\{ x \cdot y \;\middle|\; x > 0, y > 0;\ x \in L_\alpha, y \in L_\beta \right\}$$
$$1 := \left\{ q \in \mathbb{Q} \;\middle|\; q < 1 \right\}$$
$$\alpha^{-1} := \left\{ q \in \mathbb{Q} \;\middle|\; q > 0, \exists\ r > 0 \text{ such that } \frac{1}{q} - r = q^{-1} - r \in U_\alpha \right\}$$

With these definitions we need to check that (*exercise*)

1. $\cdot$ is commutative and associative
2. 1 is indeed the identity of the product, i.e. $1 \cdot \alpha = \alpha\ \forall\ \alpha \in \mathbb{R}$
3. α^{-1} is the inverse of α for the product, i.e. $\alpha \cdot \alpha^{-1} = 1\ \forall\ \alpha \in \mathbb{R}$.

Additionally, we need to check the compatibility with the ordering relation:

$$\forall\ \alpha, \beta \in \mathbb{R} \text{ such that } \alpha, \beta > 0, \text{ then } \alpha \cdot \beta > 0.$$

Now we extend the product rule to the 0 and the negative α's by setting

$$0 \cdot \alpha = 0 \qquad \forall\, \alpha \in \mathbb{R}$$

and

$$\alpha \cdot \beta = \begin{cases} (-\alpha) \cdot (-\beta) & \alpha < 0, \beta < 0 \\ -\left[(-\alpha) \cdot \beta\right] & \alpha < 0, \beta > 0 \\ -\left[\alpha \cdot (-\beta)\right] & \alpha > 0, \beta < 0 \end{cases}$$

And we check the distributive rule.

Conclusion. $(\mathbb{R}, +, \cdot; <)$ is an ordered field.

Step 4

$\mathbb{R}$ has the least-upper-bound-property, i.e. for any subset $S \subseteq \mathbb{R}$ bounded above there exists $\sup S \in \mathbb{R}$.

Proof. Set

$$\gamma = (L_\gamma, U_\gamma) := \left(\bigcup_{\alpha \in S} L_\alpha, \bigcap_{\alpha \in S} U_\alpha \right),$$

then $\gamma \in \mathbb{R}$ (i.e. γ is a Dedekind cut) because

(a) L_α has no maximum for all α's, therefore $L_\gamma := \bigcup_{\alpha \in S} L_\alpha$ doesn't have maximum either

(b) $\forall\, x \in L_\gamma$, $y \in U_\gamma$, we have that

$$x \in L_\gamma = \bigcup_{\alpha \in S} L_\alpha \quad \Rightarrow \quad \exists\, \bar{\alpha} \in S \text{ such that } x \in L_{\bar{\alpha}}$$

and

$$y \in U_\gamma = \bigcap_{\alpha \in S} U_\alpha \quad \Rightarrow \quad y \in U_\alpha \;\forall\, \alpha \in S, \text{ in particular } y \in U_{\bar{\alpha}}$$

therefore the couple (x, y) belongs to the Dedekind cut $\bar{\alpha} = (L_{\bar{\alpha}}, U_{\bar{\alpha}})$, meaning $x < y$.

γ is an upper bound for S, because $\forall\, \alpha \in S$ we have $L_\alpha \subseteq \bigcup_{\alpha \in S} L_\alpha = L_\gamma$, therefore $\alpha \leq \gamma$.

Furthermore, γ is the least upper bound: consider any other upper bound $\tilde{\gamma}$ for S, then $\alpha \leq \tilde{\gamma}$ $\forall\, \alpha \in S$, meaning $L_\alpha \subseteq L_{\tilde{\gamma}}$. Taking the union on the left hand side over all the α's we get: $L_\gamma = \bigcup_{\alpha \in S} L_\alpha \subseteq L_{\tilde{\gamma}}$, implying $\gamma \leq \tilde{\gamma}$ (i.e. γ is the smallest upper bound of all). $\square$

Conclusion. We found what we wanted: $(\mathbb{R}, +, \cdot; <)$ **is a complete** (i.e. it has the least-upper-bound property) **ordered field containing** $\mathbb{Q}$.

Last note. Is $\mathbb{R}$ the only field with such properties that we can construct? The short answer (*unproved for now*) is that $\mathbb{R}$ is the unique field with all these properties, up to isomorphism (a continuous bijection that preserves the sum, product and ordering of the elements).