Notes on Introduction on Fourier Series

Manuela Girotti

MATH 317-01 Advanced Calculus of one variable

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1 Introduction and motivations

In the early 1800s Joseph Fourier developed a new type of series (that will later on take his name) in his famous treatise on heat flow. We will give here a quick introduction of this very wide theory that is Fourier Analysis.

Definition 1. A function $f : [a, b] \to \mathbb{R}$ is said **piece-wise continuous** if it is continuous on $[a, b]$ except on a finite number of points $a = x_0 < x_2 < \ldots < x_n = b$ where

$$\lim_{x \to (x_i)_+} f(x) \qquad \text{and} \qquad \lim_{x \to (x_{i-1})_-} f(x)$$

exist for all $i = 1, \ldots, n$.

Definition 2. A function $f : \mathbb{R} \to \mathbb{R}$ is **periodic** with period $P > 0$ if $f(x + P) = f(x)$ for all $x \in \mathbb{R}$.

Examples. The function $f(x) = \sin(x)$ or the function $g(x) = \cos(x)$ are both periodic function with period $P = 2\pi$. The function $h(x) = \tan(x)$ is a periodic function with period $P = \pi$.

Definition 3. Given a positive function $q : [a, b] \to \mathbb{R}_+$ (we call it **weight function**), two functions $f, g : [a, b] \to \mathbb{R}$ are said to be **orthogonal with respect to the weight** q if

$$\int_a^b f(x)g(x)\, q(x)\mathrm{d}x = 0$$

For a given (positive) function q, it is often possible to find an infinite sequence of functions $\{\phi_n(x)\}_{n=0}^{\infty}$ such that they are all mutually orthogonal between each other

$$\int_a^b \phi_n(x)\phi_m(x)\, q(x)\mathrm{d}x = 0 \qquad \text{if } m \neq n.$$

If such a sequence exists, then it is called an **orthogonal system** of functions. Suppose that we are also imposing that

$$0 < \int_a^b \phi_n(x)\phi_n(x)\, q(x)\mathrm{d}x = \int_a^b \phi_n(x)^2\, q(x)\mathrm{d}x \overset{!}{=} 1$$

then the orthogonal system is called an **orthonormal system**.

Example 1. The sequence of functions $\{\phi_n(x) = \sin(nx)\}_{n=0}^{\infty}$ is an orthogonal system on the interval $[0,\pi]$ with respect to the weight $q(x) = 1$: indeed, for any $m, n \in \mathbb{N}$ ($m \neq n$)

$$\begin{aligned}\int_0^\pi \sin(nx)\sin(mx)\mathrm{d}x &= \frac{1}{2}\int_0^\pi \left[\cos((m-n)x) - \cos((m+n)x)\right]\mathrm{d}x = \\ &= \frac{1}{2}\left[\frac{\sin((m-n)x)}{m-n} - \frac{\sin((m+n)x)}{m+n}\right]_0^\pi = 0\end{aligned}$$

Also, the sequence $\left\{\sqrt{\frac{2}{\pi}}\sin(nx)\right\}_{n\geq 0}$ is an orthonormal system on $[0,\pi]$ with respect to the weight $q(x) = 1$.

Exercise: prove that this sequence is indeed an orthonormal system.

Remark 4. *Given a (piece-wise continuous) function $f : [a,b] \to \mathbb{R}$, it is always possible to define a new function $\tilde{f} : \mathbb{R} \to \mathbb{R}$ that is piece-wise continuous and periodic with period $P = b - a$ ($\tilde{f}$ is called* **periodic extension** *of f). This result is easily achievable by just "glueing" several copies of f next to each other until it covers the whole real line.*

Remark 5. *If $f : \mathbb{R} \to \mathbb{R}$ is a periodic function of period $P > 0$, then the function $g : \mathbb{R} \to \mathbb{R}$ defined as $g(x) = f\left(\frac{P}{S}x\right)$ is periodic with period $S > 0$.*

From now on, we will just consider periodic functions with period $P = 2\pi$ and the function f can be considered to be defined on the interval $[a,b] = [-\pi,\pi]$ and extended periodically to the whole real line.

Example 2 [Trigonometric system]. The sequence of functions

$$\{1, \cos(x), \cos(2x), \cos(3x), \ldots, \sin(x), \sin(2x), \sin(3x), \ldots\}$$

is an orthogonal system on $[-\pi,\pi]$ with respect to the weight $q(x) = 1$.

Any finite combination of elements of this sequence

$$T_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N}\left[a_n\cos(nx) + b_n\sin(nx)\right]$$

is called a **trigonometric polynomial** of degree N.

Remark 6. *Clearly, $T_N : \mathbb{R} \to \mathbb{R}$ is a continuous and periodic function over the whole real line with period $P = 2\pi$.*

Now we can pose some "reverse" questions:

1. If $f : \mathbb{R} \to \mathbb{R}$ is a 2π-periodic function, can we express it or approximate it as a trigonometric polynomial of degree N for some $N \in \mathbb{N}$? This would mean

$$f(x) = T_N(x) + \{\text{error term}\} = \frac{a_0}{2} + \sum_{n=1}^{N} [a_n \cos(nx) + b_n \sin(nx)] + \{\text{error term}\}$$

   for some coefficients $a_0, \{a_n\}, \{b_n\}$.

   Remark 7. *The setting looks similar to the one for the Taylor polynomials.*

2. Also, if this is the case, what coefficients should we use?

3. Even more generally: suppose that the approximation gets better and better as we are taking the degree of the polynomial bigger and bigger ($N \to +\infty$). What can we say about this object:

$$S_F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)]?$$

   Does the series converges? For which $x \in \mathbb{R}$?

   Would it be true that given a 2π-periodic function f we have the equality

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)]?$$

2 Fourier Series

Let's start with considering a piece-wise continuous function $f : [-\pi, \pi] \to \mathbb{R}$ and let's assume that indeed the equality

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)]$$

makes sense, meaning that there exists some coefficients a_0, $\{a_n\}$ and $\{b_n\}$ such that f can be written as a **trigonometric series**.

To find the coefficients explicitly, we use the property that the set of functions $\{1\} \cup \{\cos(nx)\}_{n=1}^{\infty} \cup \{\sin(nx)\}_{n=1}^{\infty}$ is an orthogonal system.

First of all we integrate the function f itself (remember that f is piece-wise continuous, therefore $f \in \mathcal{R}([-\pi, \pi])$):

$$\int_{-\pi}^{\pi} f(x)\mathrm{d}x = \int_{-\pi}^{\pi} \left[\frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)]\right] \mathrm{d}x$$

$$\stackrel{\star}{=} \frac{a_0}{2} \cdot 2\pi + \sum_{n=1}^{\infty} \left[a_n \int_{-\pi}^{\pi} \cos(nx)\mathrm{d}x + b_n \int_{-\pi}^{\pi} \sin(nx)\mathrm{d}x\right] = a_0\pi + \sum_{n=1}^{\infty} \left[a_n \left[\frac{\sin(nx)}{n}\right]_{-\pi}^{\pi} + b_n \left[\frac{-\cos(nx)}{n}\right]_{-\pi}^{\pi}\right]$$

$$= a_0\pi$$

Therefore,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\mathrm{d}x$$

We integrate now the product of the function f with any other element of our trigonometric (orthogonal) system: for any $k \in \mathbb{N}$,

$$\int_{-\pi}^{\pi} f(x)\sin(kx)\mathrm{d}x \overset{\star}{=} \frac{a_0}{2}\cancel{\int_{-\pi}^{\pi}\sin(kx)\mathrm{d}x} + \sum_{n=1}^{\infty}\left[a_n \cancel{\int_{-\pi}^{\pi}\cos(nx)\sin(kx)\mathrm{d}x} + b_n\int_{-\pi}^{\pi}\sin(nx)\sin(kx)\mathrm{d}x\right]$$
$$= b_k\int_{-\pi}^{\pi}(\sin(kx))^2\,\mathrm{d}x = b_k\int_{-\pi}^{\pi}\frac{1-\cos(2kx)}{2}\mathrm{d}x = \frac{b_k}{2}\left[2\pi - \frac{\sin(2kx)}{2k}\right]_{-\pi}^{\pi} = b_k\pi$$

and

$$\int_{-\pi}^{\pi} f(x)\cos(kx)\mathrm{d}x \overset{\star}{=} \frac{a_0}{2}\cancel{\int_{-\pi}^{\pi}\cos(kx)\mathrm{d}x} + \sum_{n=1}^{\infty}\left[a_n \int_{-\pi}^{\pi}\cos(nx)\cos(kx)\mathrm{d}x + b_n\cancel{\int_{-\pi}^{\pi}\sin(nx)\cos(kx)\mathrm{d}x}\right]$$
$$= a_k\int_{-\pi}^{\pi}(\cos(kx))^2\,\mathrm{d}x = a_k\pi$$

implying

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)\mathrm{d}x \qquad \text{and} \qquad b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)\mathrm{d}x$$

Note 8. *The equalities $\overset{\star}{=}$ is not automatic and it requires some theorems to be justified; in general you need to prove that some conditions hold so that you can swap the integral with the series.*

Definition 9. Given a function $f : [-\pi, \pi] \to \mathbb{R}$, piece-wise continuous, the coefficients

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\mathrm{d}x, \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\mathrm{d}x \qquad \text{and} \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\mathrm{d}x$$

are called **Fourier coefficients** of the function f.

The trigonometric series built out combining the above coefficients and the trigonometric system

$$S_F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos(nx) + b_n\sin(nx)\right]$$

is called **Fourier series** associated to the function f.

Remark 10. *We can notice that since f is piece-wise continuous on $[-\pi, \pi]$, then $f \in \mathcal{R}([\pi, \pi])$ and therefore the Fourier coefficients are well-defined ($f \cdot \cos$, $f \cdot \sin \in \mathcal{R}([\pi, \pi])$).*

Example 1. Consider the function

$$f(x) = \begin{cases} -1 & x \in [-\pi, 0) \\ 1 & x \in [0, \pi] \end{cases}$$

and calculate its Fourier series.

Then,

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\mathrm{d}x = \int_{\pi}^{0}(-1)\mathrm{d}x + \int_{0}^{\pi}(1)\mathrm{d}x = 0 \\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\mathrm{d}x = \int_{\pi}^{0} -\cos(nx)\mathrm{d}x + \int_{0}^{\pi} \cos(nx)\mathrm{d}x = 0 \\
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\mathrm{d}x = \int_{\pi}^{0} -\sin(nx)\mathrm{d}x + \int_{0}^{\pi} \sin(nx)\mathrm{d}x \\
&= \frac{1}{\pi}\left[\frac{\cos(nx)}{n}\right]_{-\pi}^{0} + \frac{1}{\pi}\left[\frac{-\cos(nx)}{n}\right]_{0}^{\pi} = \frac{2}{n\pi}\left(1-\cos(n\pi)\right) = \begin{cases} 0 & n = 2k \text{ (even)} \\ \frac{4}{\pi(2k-1)} & n = 2k-1 \text{ (odd)} \end{cases}
\end{aligned}$$

Therefore, the Fourier series associated to the function f above is equal to

$$S_F(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin\left((2n-1)x\right).$$

Example 2. Consider the function $f(x) = x^2$ restricted over the interval $[-\pi, \pi]$ and calculate its Fourier series.

Then,

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\mathrm{d}x = \frac{2}{\pi}\frac{\pi^3}{3} = \frac{2\pi^2}{3} \\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos(nx)\mathrm{d}x = \ldots\text{by parts}\ldots = \frac{1}{\pi}\left(\cancel{\left[x^2\frac{\sin(nx)}{n}\right]_{-\pi}^{\pi}} + \frac{2}{n^2}\left[x\cos(nx)\right]_{-\pi}^{\pi} - \cancel{\frac{2}{n^2}\int_{-\pi}^{\pi}\cos(nx)\mathrm{d}x}\right) \\
&= \frac{2}{n^2}\cdot 2\pi\cos(n\pi) = \frac{(-1)^n 4}{n^2} \\
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\sin(nx)\mathrm{d}x = 0
\end{aligned}$$

Therefore, the Fourier series associated to the function f above is equal to

$$S_F(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(nx).$$

Remark 11. *Assume that we have the equality $f(x) = S_F(x)$ for $x \in [-\pi, \pi]$ (this is indeed the case; we will get there shortly), i.e.*

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(nx),$$

and evaluate the function at the endpoint $x = \pi$:

$$\pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(n\pi) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}(-1)^n = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{1}{n^2}$$

meaning

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Proposition 12. *If a function is odd (i.e. $f(-x) = -f(x)$), then the coefficients $a_n = 0$ for all $n \in \mathbb{N} \cup \{0\}$.*

If a function is even (i.e. $f(-x) = f(x)$), then the coefficients $b_n = 0$ for all $n \in \mathbb{N}$.

2.1 Fourier series over any interval

In general, Fourier series (with sine and cosine) can be defined over any interval $[\alpha, \beta]$. Let us consider a function $f(t)$ periodic with period 2π over the interval $[-\pi, \pi]$, then its Fourier series is given as before

$$S_F(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(nt) + b_n \sin(nt)]$$

with

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(nt) \mathrm{d}t \qquad \text{and} \qquad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(nt) \mathrm{d}t.$$

Let $\alpha, \beta \in \mathbb{R}$ (suppose $\alpha < \beta$) and define a new variable

$$x = \frac{\beta - \alpha}{2\pi} t + \frac{\beta + \alpha}{2}, \qquad t \in [-\pi, \pi]$$

or viceversa $t = \frac{\pi}{\beta - \alpha}(2x - \beta - \alpha)$ ($x \in [\alpha, \beta]$). We can then perform a change of variable and define a new function $g : [\alpha, \beta] \to \mathbb{R}$ as

$$g(x) = f(t)\Big|_{t = \frac{\pi}{\beta-\alpha}(2x-\beta-\alpha)} = f\left(\frac{\pi}{\beta - \alpha}(2x - \beta - \alpha)\right).$$

Then, the Fourier transform of g is

$$S_F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi}{\beta - \alpha}(2x - \beta - \alpha)\right) + b_n \sin\left(\frac{n\pi}{\beta - \alpha}(2x - \beta - \alpha)\right)\right]$$

with

$$a_n = \frac{2}{\beta - \alpha} \int_{\alpha}^{\beta} g(x) \cos\left(\frac{n\pi}{\beta - \alpha}(2x - \beta - \alpha)\right) \mathrm{d}x$$
$$b_n = \frac{2}{\beta - \alpha} \int_{\alpha}^{\beta} g(x) \sin\left(\frac{n\pi}{\beta - \alpha}(2x - \beta - \alpha)\right) \mathrm{d}x.$$

3 Point-wise convergence

Theorem 13 (Point-wise convergence). *If f is a periodic function with period $P = 2\pi$ and both f and f' are piece-wise continuous on $[-\pi, \pi]$, then the Fourier series*

$$S_F(x) = \frac{a_0}{2} + \sum_{n=1}^{+\infty} [a_n \cos(nx) + b_n \sin(nx)]$$

is convergent. Moreover, for any $x \in [-\pi, \pi]$ where the function is continuous, we have the equality

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{+\infty} [a_n \cos(nx) + b_n \sin(nx)]$$

while for the points $x \in [-\pi, \pi]$ where the function is discontinuous (remember that we only have a finite number of them and the left- and righ-side limits exist), we have

$$\frac{f(x^-) + f(x^+)}{2} = \frac{a_0}{2} + \sum_{n=1}^{+\infty} [a_n \cos(nx) + b_n \sin(nx)]$$

Proof. The proof consists of showing that the sequence of partial sums is convergent and indeed it converges to the value of the function f at the point x (or the average of the left- and right-side limit). To achieve the result, we will use three Lemmas that will be stated and proved along the way.

Lemma 14. *For all $n \in \mathbb{N} \cup \{0\}$, we have*

$$a_n \cos(nx) + b_n \sin(nx) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) \cos(nt) \mathrm{d}t$$
$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x-t) \mathrm{d}t$$

Proof. It follows from the definition of the Fourier coefficients plus some smart manipulations:

$$a_n \cos(nx) + b_n \sin(nx) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \left[\cos(nt)\cos(nx) + \sin(nt)\sin(nx)\right] \mathrm{d}t$$
$$= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(n(t-x)) \mathrm{d}t = \frac{1}{\pi} \int_{-(\pi-x)}^{\pi-x} f(s+x) \cos(ns) \mathrm{d}s = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+s) \cos(ns) \mathrm{d}s$$

where in the last equality we used the fact that if a function g has period $P = 2\pi$, then $\int_{\alpha}^{\alpha+2\pi} g(t)\mathrm{d}t = \int_{-\pi}^{\pi} g(t)\mathrm{d}t$ for any $\alpha \in \mathbb{R}$. The same holds for the coefficients a_0. □

Let's consider now the sequence of partial sums:

$$S_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} [a_n \cos(nx) + b_n \sin(nx)]$$
$$= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) \left(\frac{1}{2} + \sum_{n=1}^{N} \cos(nt) \right) \mathrm{d}t = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) D_N(t) \mathrm{d}t$$

by Lemma 14. We introduced here a new function called Dirichlet's kernel

$$D_N(x) := \frac{1}{2} + \sum_{n=1}^{N} \cos(nt)$$

which has the following property:

Lemma 15.

$$D_N(x) = \frac{\sin\left((N+\frac{1}{2})x\right)}{2\sin\left(\frac{x}{2}\right)} \qquad \forall\ x \in \mathbb{R}.$$

Furthermore,

$$\int_0^{\pi} D_N(x)\mathrm{d}x = \frac{\pi}{2}, \qquad \int_{-\pi}^{0} D_N(x)\mathrm{d}x = \frac{\pi}{2}.$$

Proof. Indeed, we rewrite the cosine function into its exponential form $\cos(x) = \frac{e^{ix}+e^{-ix}}{2}$ (there's a similar one for the sine function) and we use a result about geometric sums, namely $\sum_{n=0}^{N} q^n = \frac{1-q^{N+1}}{1-q}$, provided that $q \neq 1$. Let's assume $x \neq 0$ (i.e. $e^{ix} \neq 1$),

$$D_N(x) = \frac{1}{2} + \sum_{n=1}^{N} \cos(nx) = \frac{1}{2}\left(1 + \sum_{n=1}^{N} e^{inx} + \sum_{n=1}^{N} e^{-inx}\right) = \frac{1}{2}\left(1 + \frac{e^{ix} - e^{i(N+1)x}}{1-e^{ix}} + \frac{e^{-ix} - e^{-i(N+1)x}}{1-e^{-ix}}\right)$$

$$= \frac{1}{2}\left(1 + \frac{e^{ix} - e^{i(N+1)x}}{e^{\frac{ix}{2}}(e^{-\frac{ix}{2}} - e^{\frac{ix}{2}})} + \frac{e^{-ix} - e^{-i(N+1)x}}{e^{-\frac{ix}{2}}(e^{\frac{ix}{2}} - e^{-\frac{ix}{2}})}\right) = \frac{1}{2}\left[1 + \frac{1}{2i\sin\left(\frac{x}{2}\right)}\left(-e^{\frac{ix}{2}} + e^{i(N+\frac{1}{2})x} + e^{-\frac{ix}{2}} - e^{-i(N+\frac{1}{2})x}\right)\right]$$

$$= \frac{1}{2}\left[1 + \frac{1}{2i\sin\left(\frac{x}{2}\right)}\left(-2i\sin\left(\frac{x}{2}\right) + 2i\sin\left(\left(N+\frac{1}{2}\right)x\right)\right)\right] = \frac{\sin\left((N+\frac{1}{2})x\right)}{2\sin\left(\frac{x}{2}\right)}.$$

This formula is now clearly valid for $x = 0$: we just need to calculate a limit: $\lim_{x\to 0} \frac{\sin\left((N+\frac{1}{2})x\right)}{2\sin\left(\frac{x}{2}\right)} = \lim_{x\to 0} \frac{\sin\left((N+\frac{1}{2})x\right)}{(N+\frac{1}{2})x} \frac{x}{2\sin\left(\frac{x}{2}\right)} \frac{(N+\frac{1}{2})x}{x} = N + \frac{1}{2}$.

The second batch of formulægiven in the Lemma is just a matter of straightforward calculations. ∎

Getting back to the sequence of partial sums and using Lemma 15, we have

$$S_N(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\frac{\sin\left((N+\frac{1}{2})t\right)}{2\sin\left(\frac{t}{2}\right)}\mathrm{d}t$$

Now, instead of proving that $S_N(x) \to \frac{f(x^+)+f(x^-)}{2}$ as $N \to +\infty$, we will prove that

$$\frac{1}{\pi}\int_{0}^{\pi} f(x+t)\frac{\sin\left((N+\frac{1}{2})t\right)}{2\sin\left(\frac{t}{2}\right)}\mathrm{d}t \longrightarrow \frac{f(x^+)}{2}$$
$$\frac{1}{\pi}\int_{-\pi}^{0} f(x+t)\frac{\sin\left((N+\frac{1}{2})t\right)}{2\sin\left(\frac{t}{2}\right)}\mathrm{d}t \longrightarrow \frac{f(x^-)}{2}.$$

We will focus only on the first limit (the second one follows the same procedure): from Lemma 15 (the second batch of formulæ),

$$\frac{1}{\pi}\int_0^{\pi} f(x+t)\frac{\sin\left((N+\frac{1}{2})t\right)}{2\sin\left(\frac{t}{2}\right)}\mathrm{d}t - \frac{f(x^+)}{2} = \frac{1}{\pi}\int_0^{\pi}\frac{f(x+t)-\frac{f(x^+)}{2}}{2\sin\left(\frac{t}{2}\right)}\sin\left((N+\frac{1}{2})t\right)\mathrm{d}t$$
$$= \frac{1}{\pi}\int_0^{\pi} g_x(t)\sin\left((N+\frac{1}{2})t\right)\mathrm{d}t$$

where

$$g_x(t) := \frac{f(x+t)-\frac{f(x^+)}{2}}{2\sin\left(\frac{t}{2}\right)}.$$

By the properties of f, g is piece-wise continuous for all $t \in (0,\pi]$. Moreover,

$$\lim_{t\to 0_+} g_x(t) = \lim_{t\to 0_+}\frac{f(x+t)-\frac{f(x^+)}{2}}{2\sin\left(\frac{t}{2}\right)} = \lim_{t\to 0_+}\frac{f(x+t)-\frac{f(x^+)}{2}}{t}\frac{t}{2\sin\left(\frac{t}{2}\right)} = D_+ f(x)$$

(the right-side derivative of f at the point x), meaning that $g_x(t)$ is also well-defined in zero and $g_x \in \mathcal{R}([0,\pi])$.

Lemma 16 (Riemann-Lebesgue Lemma). *Let $f \in \mathcal{R}([a,b])$, then as $\lambda \to +\infty$*

$$\frac{1}{\pi}\int_0^{\pi} f(t)\sin(\lambda t)\,\mathrm{d}t \to 0 \qquad \textit{and} \qquad \frac{1}{\pi}\int_0^{\pi} f(t)\cos(\lambda t)\,\mathrm{d}t \to 0.$$

Proof. If $f(t) = C$ a constant function, then it is obvious: $\left|\frac{1}{\pi}\int_0^{\pi} f(t)\sin(\lambda t)\,\mathrm{d}t\right| = \left|\frac{C}{\pi}\left[\frac{-\cos(\lambda t)}{\lambda}\right]_a^b\right| \leq \frac{2|C|}{\pi\lambda} \to 0.$

If $f(x) = \sum_{k=1}^{K} C_k|_{[x_{k-1},x_k]}$ is piecewise constant (where $\{x_k\}_0^K$ is a partition of $[a,b]$), then the same principle holds and we have $\left|\frac{1}{\pi}\int_0^{\pi} f(t)\sin(\lambda t)\,\mathrm{d}t\right| \leq \frac{\sum_{k=1}^{K} 2|C_k|}{\pi\lambda} \to 0.$

For a generic function that is Riemann-integrable $f \in \mathcal{R}([a,b])$, we know that we can find a partition P that can approximate the value of the integral arbitrarily well: let $\epsilon > 0$ and call $g(t)$ the piecewise constant function that describes the lower sums of f with the partition P, then

$$0 \leq \frac{1}{\pi}\int_0^{\pi} f(t)\mathrm{d}t - L(f,P) = \frac{1}{\pi}\int_0^{\pi} f(t)\mathrm{d}t - \frac{1}{\pi}\int_a^b g(t)\mathrm{d}t = \frac{1}{\pi}\int_0^{\pi}\left(f(t)-g(t)\right)\mathrm{d}t < \epsilon$$

(by construction $f(t) - g(t)$ is a non-negative function).

In conclusion,

$$\left|\frac{1}{\pi}\int_0^{\pi} f(t)\sin(\lambda t)\,\mathrm{d}t\right| \leq \left|\frac{1}{\pi}\int_0^{\pi}\left(f(t)-g(t)\right)\sin(\lambda t)\,\mathrm{d}t\right| + \left|\frac{1}{\pi}\int_0^{\pi} g(t)\sin(\lambda t)\,\mathrm{d}t\right|$$
$$\leq \frac{1}{\pi}\int_0^{\pi}|f(t)-g(t)|\,\mathrm{d}t + \frac{\mathcal{K}}{\lambda} < \epsilon + \frac{\mathcal{K}}{\lambda} \to 0$$

(we take ϵ smaller and smaller).

The same arguments hold for the "cos" version. ☐

Finally, thanks to Lemma 16, as $N \to +\infty$ we have

$$\frac{1}{\pi}\int_0^\pi f(x+t)\frac{\sin\left((N+\frac{1}{2})t\right)}{2\sin\left(\frac{t}{2}\right)}\mathrm{d}t - \frac{f(x^+)}{2} = \frac{1}{\pi}\int_0^\pi g_x(t)\sin\left((N+\frac{1}{2})t\right)\mathrm{d}t \to 0.$$

The same holds for the convergence to $\frac{f(x^-)}{2}$. □

4 The Gibb's phenomenon

We start with the remark that a function f has a jump discontinuity of amplitude b at the point $x = c$ if

$$\lim_{\epsilon\to 0;\,\epsilon>0} |f(c-\epsilon) - f(c+\epsilon)| = b$$

Viceversa, f is continuous at $x = c$ if the limit above equals zero.

We will see that if a periodic function f is discontinuous, then its Fourier series behaves in a strange way.

The behaviour is called **Gibbs' phenonemon** and it says that the truncated Fourier series (i.e. the Fourier trigonometric polynomial)

$$T_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N}\left[a_n\cos(nx) + b_n\sin(nx)\right]$$

near a jump discontinuity exceeds the jump by about 9% of the size of the jump, no matter how big the order N of the polynomial is. This means that the entire Fourier series doesn't match the function very well in a neighbourhood of the discontinuity (not only at the discontinuity point itself, where we know that the value of the Fourier series is equal to $\frac{f(x_-)+f(x_+)}{2}$, thanks to the theorem above).

To study this phenomenon we will consider one simple example. Consider the "square wave" function that we saw in Example 1:

$$f : [-\pi, \pi] \to \mathbb{R}$$
$$f(x) = \begin{cases} -1 & x \in [-\pi, 0) \\ 1 & x \in [0, \pi] \end{cases}$$

periodically extended over the whole real line.

Since the jump discontinuity at $x = 0$ is equal to $b = 1 + (-1) = 2$, we will see that the peak value of the Fourier series is about 0.18 (i.e. 9% of the value 2) higher than the maximum value of the function f at the discontinuity point $x = 0$.

We already know its Fourier series:

$$S_F(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin\left((2n-1)x\right)}{2n-1};$$

and its truncated Fourier series (i.e. the Fourier trigonometric polynomial of degree N):

$$T_N(x) = \frac{4}{\pi}\sum_{n=1}^{N}\frac{\sin\left((2n-1)x\right)}{2n-1} = \frac{4}{\pi}\left(\sin(x) + \frac{\sin(3x)}{3} + \ldots + \frac{\sin\left((2N-1)x\right)}{2N-1}\right).$$

Proposition 17. *For all $x \in \mathbb{R}$*

$$[T_N(x)]' = \frac{2}{\pi}\frac{\sin(2Nx)}{\sin(x)}.$$

Proof. We first take the derivative of the truncated Fourier series from the formula above

$$[T_N(x)]' = \frac{4}{\pi}\left(\cos(x) + \cos(3x) + \ldots + \cos\left((2N-1)x\right)\right)$$

and we use the equivalent expressions for sine and cosine functions ($\cos(x) = \frac{e^{ix}+e^{-ix}}{2}$ and $\sin(x) = \frac{e^{ix}-e^{-ix}}{2i}$) to get

$$\begin{aligned}
[T_N(x)]' &= \frac{4}{\pi}\left(\frac{e^{ix}+e^{-ix}}{2} + \frac{e^{3ix}+e^{-3ix}}{2} + \ldots \frac{e^{(2N-3)ix}+e^{-(2N-3)ix}}{2} + \frac{e^{(2N-1)ix}+e^{-(2N-1)ix}}{2}\right) \\
&= \frac{2}{\pi}e^{-(2N-1)ix}\left(1 + e^{2ix} + e^{4ix} + \ldots + e^{(4N-4)ix} + e^{(4N-2)ix}\right) = \frac{2}{\pi}e^{-(2N-1)ix}\sum_{n=0}^{4N-2}\left(e^{2ix}\right)^n
\end{aligned}$$

this is a geometric sum with general term $q = e^{2ix}$ (and $|q| = |e^{2ix}| < 1$), therefore its sum is equal to

$$\begin{aligned}
= \frac{2}{\pi}e^{-(2N-1)ix}\frac{1-e^{(4N-2)ix}}{1-e^{2ix}} &= \frac{1}{\pi}\frac{e^{-(2N-1)ix}-e^{(2N-1)ix}}{ie^{ix}\frac{(e^{-ix}-e^{ix})}{2i}} = \frac{1}{\pi}\frac{2e^{ix}\frac{e^{-2Nix}-e^{2Nix}}{2i}}{e^{ix}(-\sin(x))} \\
&= \frac{2}{\pi}\frac{\sin(2Nx)}{\sin(x)}
\end{aligned}$$

□

In order to find the maximum value(s) of the function $T_N(x)$ we study the zeroes of the derivative and we can clearly see that the first zero of the derivative is for $x = \frac{\pi}{2N}$. Since $T_N(0) = 0$ and the terms in the sum for $T_N(\frac{\pi}{2N})$ are all positive, we can conclude that $x = \frac{\pi}{2N}$ is a maximum (it's actually a global maximum).

We know that $f(\frac{\pi}{2N}) = 1$ (since $x = \frac{\pi}{2N} \in [0, \pi]$) and we want to calculate (or better, to estimate) what is the value of the trigonometric polynomial $T_N(x)$ at the point $x = \frac{\pi}{2N}$.

Remember that $T_N(x)$ should approximate the "square-wave" function $f(x)$ and eventually should be equal to $f(x)$ when $N \to +\infty$ (i.e. when we get the full Fourier series and not just a truncation of it).

$$\begin{aligned}
T_N\left(\frac{\pi}{2N}\right) &= \frac{4}{\pi}\left(\sin\left(\frac{\pi}{2N}\right) + \frac{\sin\left(\frac{3\pi}{2N}\right)}{3} + \ldots + \frac{\sin\left(\frac{(2N-1)\pi}{2N}\right)}{2N-1}\right) \\
= \frac{4}{\pi}\frac{\pi}{2N}&\left(\frac{\sin\left(\frac{\pi}{2N}\right)}{\frac{\pi}{2N}} + \frac{\sin\left(\frac{3\pi}{2N}\right)}{\frac{3\pi}{2N}} + \ldots + \frac{\sin\left(\frac{(2N-1)\pi}{2N}\right)}{\frac{(2N-1)\pi}{2N}}\right) = \frac{2}{\pi}\sum_{j=0}^{N} g\left(x_{\text{mid, j}}\right)\Delta x
\end{aligned}$$

the last expression is the Riemann sum using the midpoints of the partition $P = \{x_0 = 0, x_1 = \frac{\pi}{N}, x_2 = \frac{2\pi}{N}, \ldots, x_{N-1} = \frac{(N-1)\pi}{N}, x_N = \pi\}$ and $\Delta x = x_j - x_{j-1} = \frac{\pi}{N}$ for the function $g(x) = \frac{\sin(x)}{x}$.

The functions is Riemann integrable $g \in \mathcal{R}([0,\pi])$ and therefore, as $N \nearrow +\infty$ (meaning, when the partition gets finer and finer) we have

$$\lim_{N\to+\infty} T_N\left(\frac{\pi}{2N}\right) = \lim_{N\to+\infty} \frac{2}{\pi}\sum_{j=0}^{N} g\left(x_{\text{mid, j}}\right)\Delta x = \frac{2}{\pi}\int_0^{\pi} \frac{\sin(x)}{x}\mathrm{d}x$$

All that?s left is to estimate is the value of the integral. For this we integrate the power series for $g(x) = \frac{\sin(x)}{x}$. We have that for all $x \in \mathbb{R}$

$$\frac{\sin(x)}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n},$$

which gives

$$\frac{2}{\pi}\int_0^{\pi} \frac{\sin(x)}{x}\mathrm{d}x = \frac{2}{\pi}\int_0^{\pi}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n}\mathrm{d}x = \frac{2}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\int_0^{\pi} x^{2n}\mathrm{d}x = \frac{2}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left[\frac{x^{2n+1}}{2n+1}\right]_0^{\pi}$$
$$= 2\left(1 - \frac{\pi^3}{3\cdot 3!} + \frac{\pi^4}{5\cdot 5!} - \frac{\pi^6}{7\cdot 7!} + \ldots\right) \approx 1.18$$

This series converges very rapidly and after five terms we have the value 1.18 correct to two decimal places.

We have seen that as N gets large the maximum value of $T_N(x)$ at $x = \frac{\pi}{2N}$ (and $\frac{\pi}{N} \to 0$) becomes 1.18, which is 9% bigger than the value of the jump of $f(x)$ at the same point in the limit $x = 0$.

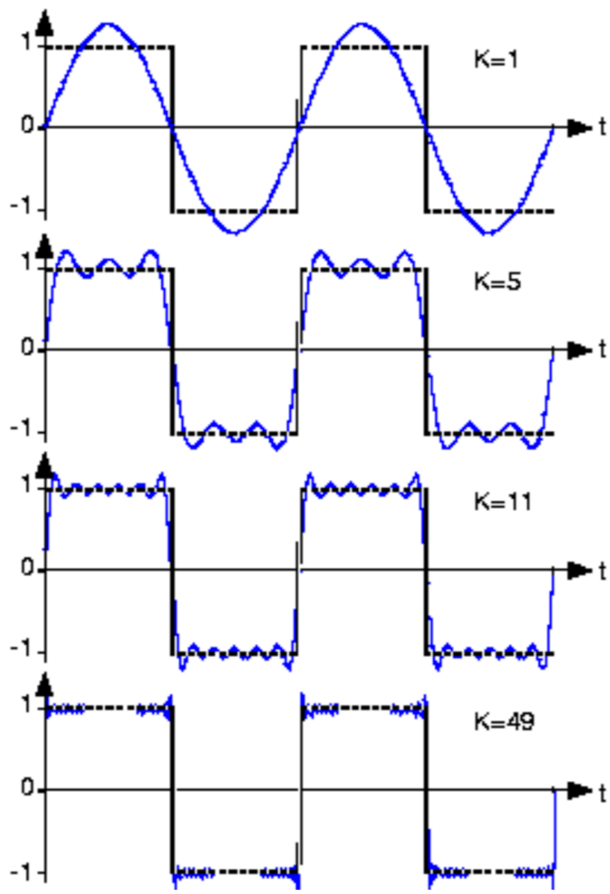

Figure 1: Fourier series approximation to the square wave function. The number of terms in the truncated Fourier sum is indicated in each plot, and the square wave is shown as a dashed line over two periods.