Using a computer to work with system of ODEs

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MATH 345 Differential Equations

Contents

We will illustrate here how to use computer packages to obtain analytical and numerical solutions for systems of ordinary differential equations. We restrict our attention to linear equations with constant coefficients.

For numerical solution, the techniques used here can be extended in an obvious way to non-linear and non-autonomous systems (see Exercise section).

1 Numerical solutions

In MATLAB®, solving systems of ODEs is not much different than solving a single (scalar) ODE. You must create a function for your system

$$\vec{x}' = \vec{F}(\vec{x}),$$

which must be then passed into a script that will use the `ode45` solver.

Example – Non-homogeneous linear system with constant coefficients.

$$\vec{x}' = \begin{bmatrix} -1 & -2 \\ 1 & -3 \end{bmatrix} \vec{x} + \begin{bmatrix} \cos(t) \\ 1 \end{bmatrix}$$

We first create a function that encodes the right-hand side of the system:

```
function [ F ] = example(t,x)
A = [−1 −2; 1 −3];

F = A*x + [cos(t); 1];

end
```

We can then numerically find the solution of the Cauchy problem with `ode45`:

```
>> t = [0:0.01:5];
>> x0 = [0 0];
>> xsol = ode45(@example, t, x0);
```

and we can plot it.

Example 2 – Phase portraits. To create phase portraits in MATLAB®, we must use the `meshgrid` command and evaluate the first derivative of x_1 and x_2 at each point (x_1, x_2) of the phase-plane, for $t = 0$.

Then, we use the `quiver` command to plot the vector field. To plot a specific solution, use the usual `ode45` command and plot the first and second columns of the output matrix.

```
>> [x,y] = meshgrid(-10:1:10, -10:1:10);

>> xprime = zeros(size(x));
>> yprime = zeros(size(x));

>> t=0;
>> for i = 1:numel(x)
        dxdt = example(t,[x(i);y(i)]);
        xprime(i) = dxdt(1);
        yprime(i) = dxdt(2);
end

>> quiver(x,y,xprime, yprime, 'b');
>> xlabel('x(t)');
>> ylabel('y(t)');

>> t = linspace(0,5);
>> x0 = [5, -7];

>> hold on
>> xsol = ode45(@(t,y)example(t,y), t, x0 );
>> plot(xsol.y(1,:), xsol.y(2,:))
>> hold off
```

2   Analytic solutions

Given a linear system

$$\vec{x}' = A\vec{x},$$

we can easily calculate the general equation by telling MATLAB® to find the eigenvalues and the eigenvectors for us and then assembling the solutions $\vec{x}_1(t), \ldots, \vec{x}_N(t)$.

Example 1 – Homogeneous linear system. Consider the linear system

$$\vec{x}' = \begin{bmatrix} -1 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \vec{x}$$

Finding the general solution with MATLAB$^{\circledR}$ is straightforward:

```
>> syms t real
>> [Evec, Eval] = eig(A);

>> X1 = exp(t*Eval(1,1)).*Evec(:,1);
etc.
```

Once a basis of linearly independent solutions is found, the general solution is $\vec{x}(t) = c_1 \vec{x}_1(t) + \ldots + c_N \vec{x}_N(t)$.

Example 2 – Matrix Exponential. Consider the same matrix as before

$$A = \begin{bmatrix} -1 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

It is possible to calculate the matrix exponential via the procedure seen in class.
Remember that given the system $\vec{x}' = A\vec{x}$, we first build the Fundamental Matrix of solutions

$$X(t) = [\vec{x}_1(t)|\vec{x}_2(t)|\ldots|\vec{x}_N(t)]$$

and the matrix exponential is then

$$e^{At} = X(t)X(0)^{-1}.$$

With MATLAB$^{\circledR}$, we have

```
>> Fund = [X1 X2 X3];
>> Fundinv0 = subs(inv(Fund), 0);

>> ExpA = Fund*Fundinv0;
```

Alternatively, we can use directly the code `expm`:

```
>> syms t real
>> ExpAdirect = expm(t*A);
```

Once we've found the matrix exponential it is straightforward to calculate the solution of the given system. Remember that given a Cauchy problem

$$\begin{cases} \vec{x}' = A\vec{x} + \vec{f}(t) \\ \vec{x}(0) = \vec{x}_0 \end{cases}$$

(if $\vec{f}(t) = 0$, then we have a standard homogeneous problem), then the solution is

$$\vec{x}(t) = e^{At}\vec{x}_0 + \int_0^t e^{A(t-s)} \vec{f}(s)\mathrm{d}s.$$

For example, for the system (the matrix is the same matrix A as in the previous example)

$$\begin{cases} \vec{x}' = \begin{bmatrix} -1 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \vec{x} + \begin{bmatrix} 1 \\ \cos(t) \\ 0 \end{bmatrix} \\ \vec{x}(0) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{cases}$$

we have

```
>> syms s real
>> xsol = ExpA*x0 + ExpA* int (expm(-s*A) * f , 0, t)
```

3   Exercises

Exercise 1 - Linear system of ODEs.   Find the general solution of the following systems and decide whether the origin is stable/asymptotically stable/unstable.

Then, compare your conclusions with the numerical results: plot the phase portrait of the system and plot some solutions with arbitrarily chosen initial condition (x_0, y_0).

1.

$$\begin{cases} x' = -2x \\ y' = x - 4y \end{cases}$$

2.

$$\begin{cases} x' = x \\ y' = -y \end{cases}$$

3.

$$\begin{cases} x' = -x \\ y' = -y \end{cases}$$

4.

$$\begin{cases} x' = -x + 2y \\ y' = -y \end{cases}$$

4

5.

$$\begin{cases} x' = 3y \\ y' = -x \end{cases}$$

6.

$$\begin{cases} x' = x - 8y \\ y' = 8x + y \end{cases}$$

Exercise 2 - Non-homogeneous linear system of ODEs.   Find the equilibrium point(s) of the following system and determine their stability properties. Compare your conclusion with the numerical results by plotting the phase portrait of the system and some solutions with arbitrarily chosen initial condition (x_0, y_0).

$$\begin{cases} x' = -2x - 6y + 8 \\ y' = 8x + 4y - 12 \end{cases}$$

Exercise 3 - Lotka-Volterra model.   Plot the phase field of the Lotka-Volterra equations

$$\begin{cases} x' = 2x - xy \\ y' = -5y + 3xy \end{cases}$$

On the same graph plot the orbits with initial conditions (x_0, y_0):

$$(1, 2), \quad (2, 1.5), \quad (0.5, 4).$$

Exercise 3 - Non-linear conservative systems.   Derive the potential energy that describes the motion of the physical system described below and deduce the phase portrait.

1. $y'' + \frac{1}{2}(y^2 - 1) = 0$

2. $y'' + 4y^3 - 4y = 0$

3. Pendulum equation: $y'' + \frac{1}{2}\sin(y) = 0$

For each of the equations, plot numerically the vector field of the phase portrait and on the same graph plot the orbits with the prescribed initial conditions.

1. $y'' + \frac{1}{2}(y^2 - 1) = 0$ with initial conditions (x_0, y_0):

$$(-4.5, 4.5), \quad (-4, 4.25), \quad (-4, 5), \quad (0, 0), \quad (1/2, 0), \quad (0.8, 0);$$

2. $y'' + 4y^3 - 4y = 0$ with initial conditions (x_0, y_0):

$$(0.1, 0.1), \quad (-0.1, 0.1), \quad (\pm 0.8, 0), \quad (0, 1), \quad (0, 2).$$

3. Pendulum equation $y'' + \frac{1}{2}\sin(y) = 0$ with initial conditions (x_0, y_0):

$$(\pm 3, 0), \quad (\pm 2, 0), \quad (-6, 0.5), \quad (6, -0.5).$$

Exercise 4 - Phase portraits for 3-dimensional systems. Plot the (3D) phase portrait of the following systems (you will need to use the command `plot3`):

1. A linear system (+ non-homogeneity):

$$\vec{x}' = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{10} \end{bmatrix} \vec{x} + \vec{f}(t)$$

with

   a) $\vec{f}(t) = \vec{0}$;
   b) $\vec{f}(t) = [\cos(t), 1, 1]^T$;
   c) $\vec{f}(t) = [\cos(x), 0, 0]^T$.

2. Lorenz system: it is a simplified mathematical model for atmospheric convection (proposed by Lorenz in 1963). A sample solution in the Lorenz attractor is when $\rho = 28$, $\sigma = 10$ and $\beta = \frac{8}{3}$.

$$\begin{cases} x' = \sigma(y - x) \\ y' = \rho x - y - xz \\ z' = xy - \beta z \end{cases}$$

3. Rössler system: it is a system of three non-linear ordinary differential equations modelling equilibrium in chemical reactions (studied Rössler in 1976). For this exercise, we set $\alpha = 0.2$, $\beta = 0.2$ and $\gamma = 5.7$.

$$\begin{cases} x' = -y - z \\ y' = x + \alpha y \\ z = \beta + z(x - \gamma) \end{cases}$$

References

[1] D.S. Jones, M.J. PLank, B.D. Sleeman, *Differential Equations and Mathematical Biology*, 2nd edition, CRC Press, 2010.

[2] Notes by A.D. Lewis, "Introduction to differential equations (for smart kids)", for the course MTHE 237 (Differential Equations for Engineering Science), Queens University, Kingston (ON), Canada, 2017.