Determinantal point processes and random matrix theory in a nutshell

– part III –

(Orthogonal Polynomials and Riemann–Hilbert method approach)

Manuela Girotti

based on M. Bertola's lectures from Les Houches Winter School 2012 and P. Miller's lecture from the thematic semester at SAMSI in 2006

Contents

Although the field of Orthogonal Polynomials is extremely vast, we will introduce here the concept of OPs purely as a tool to compute meaningful quantities coming from the Random Matrix world.

The goal will be to establish a powerful connection between the eigenvalue statistics of the unitary ensemble (represented by the kernel $K(x,y)$) and the theory of OPs.

Recap: consider the Unitary Ensemble, i.e. the set of Hermitian matrices with probability distribution

$$\mathrm{d}\mu(M) = \frac{1}{Z_n} e^{-\Lambda \operatorname{Tr}(V(M))} \mathrm{d}M$$

where we inserted now a scaling parameter Λ and we will actually take it to be exactly n, the dimension of the matrices in the ensemble (more generally, one can take $\Lambda = \frac{n}{T}$ for some $T > 0$).

We saw that the induced joint probability distribution of the eigenvalues of this ensemble is

$$\mathrm{d}\mu(x_1, \ldots, x_n) = \frac{1}{Z_n} \Delta(x_1, \ldots x_n)^2 \prod_{j=1}^{n} e^{-nV(x_j)} \mathrm{d}x_1 \ldots \mathrm{d}x_n \tag{1}$$

with $Z_n = \int_{\mathbb{R}^n} \mathrm{d}\mu(x_1, \ldots, x_n)$ a suitable normalization constant (*partition function*) and a *potential* $V(x)$ sufficiently smooth and growing sufficiently fast at infinity.

For the rest of the notes, we shall choose $V(x)$ to be a polynomial of even degree, with positive leading coefficient (e.g. $V(x) = x^2$).

We also saw that the jpdf can be crucially rewritten in a determinantal form:

$$\frac{1}{Z_n} \prod_{1 \leq i < j \leq n} (x_1, \ldots x_n)^2 \prod_{i=1}^{n} e^{-nV(x_i)} = \frac{1}{n!} \det \Big[K_n(x_i, x_j) \Big]_{1 \leq i,j \leq n} \tag{2}$$

where

$$K(x, y) = e^{-\frac{n}{2}(V(x)+V(y))} \sum_{j,k=0}^{n-1} x^j [\mathbb{M}]^{-1}_{jk} y^k \tag{3}$$

and the matrix $\mathbb{M}$ has entries

$$\mathbb{M}_{ab} = \int_{\mathbb{R}^n} x^{a+b} e^{-nV(x)} \mathrm{d}x \qquad 0 \leq a, b \leq n-1. \tag{4}$$

1 Orthogonal polynomials

It can be shown that $\mathbb{M}$ is (for any size) positive definite (and symmetric). Consider the Lower-Diagonal-Upper decomposition (keeping into account the symmetry)

$$\mathbb{M} = LHL^T$$

where L is a lower unipotent matrix (with ones on the diagonal) and $H = \mathrm{diag}\{h_1, \ldots, h_n\}$, then

$$K(x, y) = e^{-\frac{n}{2}(V(x)+V(y))} \begin{bmatrix} 1 & x & \ldots & x^{n-1} \end{bmatrix} L^{-T} H^{-1} L^{-1} \begin{bmatrix} 1 & y & \ldots & y^{n-1} \end{bmatrix}^T \tag{5}$$

Definition 1. The polynomials

$$\begin{bmatrix} p_0(x) \\ p_1(x) \\ \vdots \\ p_{n-1}(x) \end{bmatrix} = L^{-1} \begin{bmatrix} 1 \\ x \\ \vdots \\ x^{n-1} \end{bmatrix} \tag{6}$$

are called **orthogonal polynomials (OPs)** for the measure $e^{-V(x)}\mathrm{d}x$.

From the above definition and using formula (5), we can rephrase the kernel as

$$K(x,y) = e^{-\frac{n}{2}(V(x)+V(y))} \sum_{j=0}^{n-1} \frac{p_j(x)p_j(y)}{h_j} \tag{7}$$

Proposition 2. *The following properties holds for the OPs $p_n(x)$ and are equivalent to the above definition:*

- $\deg p_n(x) = n$ *and* $p_n(x) = x^n + \dots$;
- $\int_{\mathbb{R}} p_n(x)p_m(x)e^{-V(x)}\mathrm{d}x = h_n\delta_{nm}$;
- $\{p_j(x)\}$ *solve a three terms recurrence relation:*

$$xp_n(x) = p_{n+1} + \alpha_n p_n(x) + \frac{h_n}{h_{n-1}} p_{n-1}(x) \qquad \forall\ n.$$

In addition we have

- $h_n > 0$;
- $Z_n = n! \det \mathbb{M} = n! \prod_{j=0}^{n-1} h_j$.

Paradigma (GUE). In the case of GUE matrices, the kernel is

$$K(x,y) = e^{-\frac{n}{4}(x^2+y^2)} \sum_{j=0}^{n-1} \frac{p_j(x)p_j(y)}{h_j}$$

where the polynomials $p_n(x)$ are (a rescaled version of) the Hermite polynomials (Hermite polynomials can be indeed defined as the set of polynomials that are orthogonal with respect to the measure $e^{-x^2}\mathrm{d}x$).

Furthermore, if we consider n non-intersecting Brownian paths $X_1(t), \dots, X_n(t)$, all starting at $x = 0$ and finishing at $x = 0$ after time $T > 0$. Their transition probability is a Gaussian

$$p_t(x,y) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}} \tag{8}$$

and thanks to Karlin-McGregor Theorem, we have that the joint probability distribution of the paths at any time $0 < t < T$ is proportional to

$$\sim \det \left[F_{j-1}(x_i) e^{-\frac{x_i^2}{2t}} \right]_{i,j=1}^{n} \det \left[G_{j-1}(x_i) e^{-\frac{x_j^2}{2(T-t)}} \right]_{i,j=1}^{n} \mathrm{d}x_1 \dots \mathrm{d}x_n \tag{9}$$

(up to the normalization constant) and F_{j-1} and G_{j-1} are polynomials of degree $j-1$ obtained by consecutive derivatives of the exponential function. We recognize here an equivalent formulation of the Hermite polynomials.

Therefore, the positions of the paths at any time $0 < t < T$ are distributed as an ensemble of GUE-eigenvalues (see Figure 1).

We can push the dependency of the kernel on OPs even further and get a simple formula for the kernel.

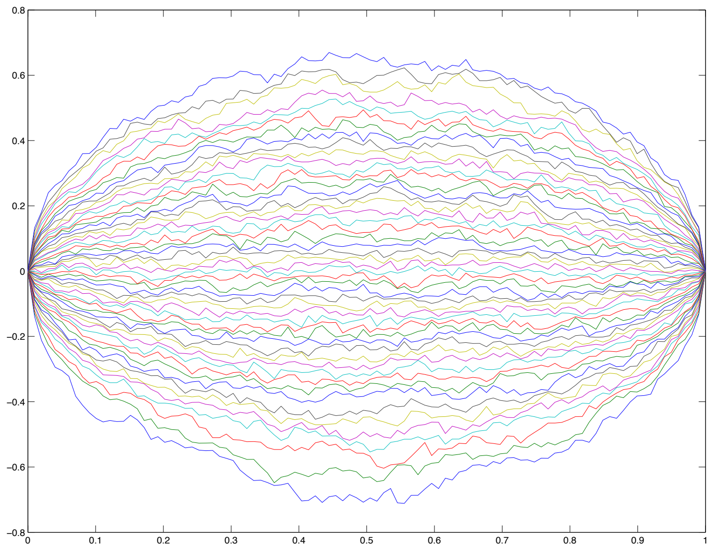

Figure 1: Numerical simulation of 50 non-intersecting Brownian paths in the confluent case with one starting and one ending point.

Proposition 3 (**Christoffel–Darboux formula**). *For any set of OPs we have*

$$K(x,y) = e^{-\frac{n}{2}(V(x)+V(y))}\frac{1}{h_n}\frac{p_n(x)p_{n-1}(y) - p_n(y)p_{n-1}(x)}{x-y} \tag{10}$$

Proof. Use the three terms recurrence relation and write it as a telescoping sum. □

2 Riemann–Hilbert problem

A Riemann–Hilbert problem is a **boundary–value problem** for a $k \times k$ matrix–valued, piecewise analytic function $Y(z)$.

Riemann–Hilbert problem 4. Let Σ be an oriented union of curves and $J(z)$ a (sufficiently smooth) matrix function defined on Σ, called the jump matrix.

Find a function $Y(z)$ such that

1. $Y(z)$ is analytic on $\mathbb{C} \setminus \Sigma$;
2. $\lim_{z\to\infty} Y(z) = 1$ (or some other normalization);
3. denoting by $Y_\pm(z)$ the (non-tangential) boundary values of $Y(z)$ from the left/right of Σ (according to the orientaiton), we have

$$Y_+(z) = Y_-(z)J(z), \qquad \forall\ z \in \Sigma$$

For the sake of simplicity, let's assume that everything is smooth enough for all the statements that follow and that the curves in Σ are either loops or they extend at infinity. We should add another condition in the case where the curves have endpoints.

In the simple case where the RHP is scalar ($k = 1$), the solution can be easily found by applying the following formula:

Theorem 5 **(Sokhotsy–Plemelji formula).** *Let $h(w)$ be α-Hölder continuous (for simplicity, we can assume h to be Lipschitz) on Σ and*

$$f(z) := \frac{1}{2\pi i} \int_{\Sigma} \frac{h(w)}{w - z} \mathrm{d}w$$

Then,

$$f_+(z) - f_-(z) = h(w)$$

and $f_+(z) + f_-(z) = H[h](w)$ exists (the Cauchy Principal value).

In the 90's Fokas, Its and Kitaev [3] proved a fundamental theorem establishing the relationship between OPs and RHPs.

Riemann–Hilbert problem 6 **(for Orthogonal Polynomials).** Find a 2×2 matrix–valued function $Y(z) = Y_n(z)$ such that

1. $Y(z)$ is analytic $\forall\ z \in \mathbb{C}_\pm = \{\pm\Im(z) > 0\}$
2. the boundary values of $Y(z)$ on $\Sigma = \mathbb{R}$ (oriented in the natural direction) are
$$Y_+(z) = Y_-(z) \begin{bmatrix} 1 & e^{-nV(z)} \\ 0 & 1 \end{bmatrix} \tag{11}$$
3. in the sectors $\arg(z) \in (0, \pi)$ and $\arg(z) \in (\pi, 2\pi)$, the matrix $Y(z)$ has the following asymptotic expansion
$$Y(z) = \left[\mathbf{1} + \mathcal{O}\left(\frac{1}{z}\right) \right] z^{n\sigma_3}, \qquad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{12}$$

The above asymptotic expansion is uniform in the sense that for any $R > 0$ there exists $C > 0$ such that for all $z \in \mathbb{C} \setminus \mathbb{R}$, $|z| > R$ we have $\|Y(z) - \mathbf{1}\| \leq C\frac{1}{|z|}$.

Before stating the solution theorem, let's assume that $V(x)$ is real-analytic and in particular let's consider the case where $V(x)$ is a polynomial of even degree and positive leading coefficient (e.g. $V(x) = x^2$).

Theorem 7 **(Fokas, Its, Kitaev).** *The unique solution to the RHP 6 is the following:*

$$Y_n(z) = \begin{bmatrix} p_n(z) & \displaystyle\int_{\mathbb{R}} \frac{p_n(x)e^{-nV(x)}}{x - z} \frac{\mathrm{d}x}{2\pi i} \\ \dfrac{-2\pi i}{h_{n-1}} p_{n-1}(z) & \dfrac{-1}{h_{n-1}} \displaystyle\int_{\mathbb{R}} \frac{p_{n-1}(x)e^{-nV(x)}}{x - z} \frac{\mathrm{d}x}{2\pi i} \end{bmatrix} \tag{13}$$

where $p_n(z)$, $p_{n-1}(z)$ are the OPs for the measure $e^{-nV(x)}\mathrm{d}x$ on $\mathbb{R}$ and h_n the corresponding squared norms.

Proof. To prove uniqueness:

1. show that $\det Y(z)$ has no jump on R (so it is an entire function);
2. show that $\det Y(z) \to 1$ as $|z| \to \infty$ and hence (by Liouville's theorem) it is identically one. Thus any solution to the RHP 6 is invertible and with analytic inverse;
3. if $\widetilde{Y}(z)$ is another solution, then show that $R(z) = \widetilde{Y}(z)Y^{-1}(z)$ has no jumps on $\mathbb{R}$ and hence it is entire;
4. using the asymptotic behaviour show that $R \to \mathbf{1}$ and hence (by Liouville's theorem again) $R(z) \equiv \mathbf{1}$.

Then, one shows directly that the proposed expression satisfies the conditions (using Sokhotsky–Plemelji's formula). ☐

Getting back to our correlation kernel $K(x,y) = K_n(x,y)$ for the eigenvalue distribution, we have

Proposition 8. *The kernel $K_n(x,y)$ can be written as*

$$K_n(x,y) = \frac{e^{-\frac{n}{2}(V(x)+V(y))}}{2\pi i(x-y)} \left[Y_\pm^{-1}(y)Y_\pm(x)\right]_{21} \tag{14}$$

3 Asymptotics

We saw previously that a good deal of interest in Random Matrix Theory is focused on studying the asymptotic behaviour of the eigenvalue statistics (in various ways). We will show here how to prove the Sine-kernel Universality in the Bulk of the eigenvalue spectrum.

Theorem 9 (Bulk universality at the origin). *For the Unitary Ensemble, the local behaviour in the bulk of the spectrum is described by a DPP with correlation kernel given by*

$$\lim_{n\to+\infty} \frac{1}{n\rho(x^*)} K_n\left(x^* + \frac{\xi}{n\rho(x^*)}, x^* + \frac{\eta}{n\rho(x^*)}\right) = \frac{\sin\left(\pi(\xi-\eta)\right)}{\pi(\xi-\eta)}. \tag{15}$$

where x^* is any point belonging to the interior of the domain of the density of eigenvalues $\rho(x)$.

Given the expression of our kernel $K_n(x,y)$ (14), it is clear that in order to find its asymptotic behaviour we need to analyze what's the asymptotic behaviour of the corresponding OPs when their degree gets large ($n \to \infty$). Going even further, it turns out that it is very convenient to study the associated RHP in the large n regime by the use of a powerful technique called non-linear steepest descent method.

For simplicity, we shall assume from now on that the potential $V(x)$ is a polynomial of even degree, positive leading coefficient and convex, but the theory holds for more general potentials.

3.1 The Steepest Descent method

The (non-linear) steepest descent method was developed in the 90's by Percy Deift and Xin Zhou [2] for mKdV and other integrable wave equations and later applied to OPs by Deift, Kriecherbauer, McLaughlin, Venakides, Zhou [1].

The method of Deift and Zhou is based upon the application of the following prototype theorem (called the Small Norm Theorem). In words, it says that if we have a RHP problem with jumps that are "close" to the identity in L^p-norm, then (as it would be natural to think) the solution itself is close to the identity matrix, where the error is explicitly given.

Theorem 10 (Small Norm Theorem). *Suppose the RHP is posed for a collection of contours Σ for a matrix $\mathcal{E}(z)$ such that*

$$\mathcal{E}_+(z) = \mathcal{E}_-(z)\big[\mathbf{1} + \delta J(z)\big] \qquad z \in \Sigma \tag{16}$$

$$\mathcal{E}(z) = \mathbf{1} + \mathcal{O}\left(\frac{1}{z}\right) \qquad \text{as } z \to \infty \tag{17}$$

with $\det(\mathbf{1} + \delta J(z)) \equiv 1$. *Then, if $\exists\, C_\Sigma \in \mathbb{R}_+$ such that*

$$\|\delta J\|_{L^\infty(\Sigma)} < C_\Sigma^{-1},$$

the solution of the RHP exists. Furthermore,

$$\|\mathcal{E}(z) - \mathbf{1}\| \leq \frac{\mathfrak{C}}{2\pi\,\mathrm{dist}(z,\Sigma)} \qquad \forall\, z \in \mathbb{C} \tag{18}$$

where $\mathfrak{C}$ is a constant depending on C_Σ and $\|\delta J\|_{L^p(\Sigma)}$ for $p = 1, 2, \infty$, and if the jump $\delta J(z)$ is analytic in a neighbourhood of Σ the denominator can be replaced by $1 + \mathrm{dist}(z, \Sigma)$.

The reason of the name is because if the norms $1, 2$ are small, then the solution $\mathcal{E}$ is close to the identity (pointwise!). In practice, the jump δJ depends on some parameter (like n in our case) and typically all L^p-norms tend to zero.

Application of the Small Norm Theorem. This is the plan for our asymptotic analysis of the RHP for OPs 6: consider the solution

$$Y_n(z) = \begin{bmatrix} p_n(z) & \displaystyle\int_{\mathbb{R}} \frac{p_n(x)e^{-nV(x)}}{x-z}\frac{\mathrm{d}x}{2\pi i} \\ \dfrac{-2\pi i}{h_{n-1}} p_{n-1}(z) & \dfrac{-1}{h_{n-1}} \displaystyle\int_{\mathbb{R}} \frac{p_{n-1}(x)e^{-nV(x)}}{x-z}\frac{\mathrm{d}x}{2\pi i} \end{bmatrix} \tag{19}$$

and suppose that we can transform it (in an invertible way) into an **equivalent** RHP with solution $W(z)$ with jump matrix $M(z)$ on some contours Σ:

$$W_+(z) = W_-(z)M(z), \quad z \in \Sigma; \qquad W(z) = \mathbf{1} + \mathcal{O}\left(\frac{1}{z}\right)., \quad z \to \infty.$$

Suppose also that we can find an **"approximate" and explicit solution** $\widetilde{W}(z)$, meaning that $\widetilde{W}$ solves a RHP whose jumps $\widetilde{M}$ are very similar to M in the sense that

$$M(z)\widetilde{M}(z)^{-1} = \mathbf{1} + \delta G(z).$$

Then, consider the "error matrix": $\mathcal{E}(z) = W(z)\widetilde{W}(z)^{-1}$. $\mathcal{E}$ solves a RHP with jumps

$$\begin{aligned}\mathcal{E}_+(z) &= W_+(z)\widetilde{W}_+(z)^{-1} = W_-(z)M(z)\widetilde{M}(z)^{-1}\widetilde{W}_-(z)^{-1} = \mathcal{E}_-(z)\widetilde{W}(z)\Big(\mathbf{1} + \delta G(z)\Big)\widetilde{W}(z)^{-1} \\ &= \mathcal{E}_-(z)\Big(\mathbf{1} + \delta J(z)\Big)\end{aligned} \tag{20}$$

If δJ satisfies (as dependent on n) the conditions of the small norm theorem, then we can rightfully consider the $\widetilde{W}$ as a good approximation of Y (up to the backward transformation $W \mapsto Y$) and the Small Norm Theorem also gives the order of approximation.

3.2 Massaging the problem: the g function

3.2.1 Repairing the normalization condition

An evident difference between the RHP (19) and a near-identity problem as described in the Small Norm Theorem is the normalization condition at infinity:

$$Y_n(z) = \begin{bmatrix} z^n & 0 \\ 0 & z^{-n} \end{bmatrix} + \{\text{ lower order terms}\dots\} \qquad \text{as } z \to \infty \tag{21}$$

which clearly states that the matrix $Y_n(z)$ does not resemble the identity matrix at least near $z = \infty$. We can try to fix this problem by cooking up a scalar function $g(z)$ that looks like $\log(z)$ near $z = \infty$. In this case, we could introduce the new function

$$W(z) = V_n(z)e^{-ng(z)\sigma_3} = V_n(z)\begin{bmatrix} e^{-ng(z)} & 0 \\ 0 & e^{ng(z)} \end{bmatrix}$$

which is close to the identity matrix in a neighbourhood of infinity: $W(z) = \mathbf{1} + \mathcal{O}\left(z^{-1}\right)$, as $z \to \infty$.

A straightforward (but naïve) choice would be to simply set $g(z) = \log(z)$. However, this way a pole of order n would appear at the origin $z = 0$, while $V_n(z)$ was analytic in the upper and lower half-planes and had continuous boundary values.

The key observation is the following: think of the function $\log(z)$ as

$$\log(z) = \int_{\mathbb{R}} \log(z - x)\, \delta_{x=0}\, \mathrm{d}x$$

where $\delta_{x=0}$ is the atomic measure supported at $x = 0$. But then we realize that any measure with total mass (integral) equal to one would have the same asymptotic behaviour at infinity.

Let's then set

$$g(z) = \int_{\mathbb{R}} \log(z - x)\, \psi(x)\, \mathrm{d}x \tag{22}$$

where $\psi(x)$ is a suitable probability density function on $\mathbb{R}$ with some nice properties, say, compactly supported and Hölder continuous for example.

This way $g(z)$ is analytic on $\mathbb{C} \setminus \mathbb{R}$ and takes continuous boundary values on $\mathbb{R}$, and moreover by expanding the integrand for large z,

$$g(z) = \int_{\mathbb{R}} \log\left(z\left(1 - \frac{x}{z}\right)\right)\, \psi(x)\, \mathrm{d}x = \log(z) + \int_{\mathbb{R}} \log\left(1 - \frac{x}{z}\right)\, \psi(x)\, \mathrm{d}x = \log(z) + \mathcal{O}\left(\frac{1}{z}\right). \tag{23}$$

We have then obtained an equivalent RHP:

Riemann–Hilbert problem 11. Find a 2×2 matrix-valued function $W(z)$ such that

1. $W(z)$ is analytic $\forall\, z \in \mathbb{C} \setminus \mathbb{R}$;
2. the boundary values are related by

$$W_+(z) = W_-(z) \begin{bmatrix} e^{-n(g_+(z)-g_-(z))} & e^{-n(V(z)-g_+(z)-g_-(z))} \\ 0 & e^{n(g_+(x)-g_-(z))} \end{bmatrix} \tag{24}$$

   where $g_\pm(z)$ denote the boundary values taken by $g(z)$ from $\mathbb{C}_\pm$;
3. $W(z) = \mathbf{1} + \mathcal{O}\left(\frac{1}{z}\right)$ for $z \to \infty$.

3.2.2 Logarithmic potential theory and equilibrium measure

How should we choose the measure $\psi(x)\mathrm{d}x$ in a smart way?

To answer the question, let's first write out more explicitly the boundary values of g. Note that for fixed $y \in \mathbb{R}$

$$(\log(x-y))_\pm = \lim_{\epsilon \searrow 0} \log(x \pm i\epsilon - y) = \begin{cases} \log|x-y| & x > y \\ \log|x-y| \pm i\pi & x < y \end{cases} \tag{25}$$

therefore, for $x \in \mathbb{R}$

$$g_\pm(x) = \int_{\mathbb{R}} \log|x-y|\, \psi(y)\, \mathrm{d}y \pm i\pi \int_x^\infty \psi(y)\, \mathrm{d}y \tag{26}$$

and

$$g_+(x) - g_-(x) = 2\pi i \int_x^\infty \psi(y)\, \mathrm{d}y \tag{27}$$

$$V(x) - g_+(x) - g_-(x) = V(x) - 2\int_{\mathbb{R}} \log|x-y|\, \psi(y)\, \mathrm{d}y \tag{28}$$

The right-hand side of (28) is the **variational derivative** or Frechét derivative of a functional of $\psi(x)\mathrm{d}x$:

$$V(x) - g_+(x) - g_-(x) = \frac{\delta E}{\delta \psi}(x) \tag{29}$$

where

$$E[\psi] = \iint_{\mathbb{R}^2} \log\left(\frac{1}{|x-y|}\right) \psi(x)\, \mathrm{d}x\, \psi(y)\, \mathrm{d}y + \int_{\mathbb{R}} V(x)\, \psi(x)\, \mathrm{d}x \tag{30}$$

Physically, $E[\psi]$ is the Coulomb energy of a distribution $\psi(x)\mathrm{d}x$ of positive electric charge confined to the real line in a two-dimensional universe (the Green's function for Laplace's equation in two dimensions is the kernel in the double integral). The charges are influenced by mutual repulsion (the logarithmic term) and by being trapped in an external electrostatic potential $V(x)$.

Theorem 12 (Gauss-Frostman, see [4]). *There is a unique probability measure $\rho(x)\mathrm{d}x$ (**equilibrium measure**) minimizing the functional $E[\psi]$.*

The minimizer $\rho(x)\mathrm{d}x$ is characterized by

$$\frac{\delta E}{\delta \psi}(x) = V(x) - g_+(x) - g_-(x) \equiv \ell \qquad x \in \operatorname{supp}\rho \tag{31}$$

$$\frac{\delta E}{\delta \psi}(x) = V(x) - g_+(x) - g_-(x) > \ell \qquad x \notin \operatorname{supp}\rho \tag{32}$$

where the constant ℓ is called Robin's constant or Lagrange multilpier.

Theorem 13 (Deift, et al.). *Suppose that $V(x)$ is also real-analytic: then* suppρ*is a finite union of compact intervals.*

Even better, if $V(x)$ is strictly convex, then there is only one interval of support:

$$V''(x) > 0 \quad \Rightarrow \quad \operatorname{supp}\rho = [\alpha, \beta].$$

Physically speaking, it seems indeed reasonable that if $V(x)$ is strictly convex then we have a "single-well" rather than a "multiple-well" potential, and we expect the charges to all be lying in a single lump.

3.2.3 Finding the g-function

We can now write out our g-function that we were originally looking for. Instead of analyzing g directly, let's consider its derivative. For $z \in \mathbb{C} \setminus \mathbb{R}$ we have

$$g'(z) = \int_{\mathbb{R}} \frac{\rho(x)\,\mathrm{d}x}{z - x} \tag{33}$$

so $g'(z)$ is a factor of $-2\pi i$ away from the Cauchy transform of ρ. If we can find $g'(z)$, then we will know $\rho(x)$ thanks to the Sokhotski-Plemelij formula

$$\rho(x) = -\frac{1}{2\pi i}\left(g'_+(x) - g'_-(x)\right) \qquad x \in \mathbb{R};$$

this also means that $g'_+(x) - g'_-(x) \equiv 0$, for $x \notin \operatorname{supp}\rho$.

On the other hand, by differentiating (31) and from the behaviour at infinity,

$$g'_+(x) + g'_-(x) = V'(x) \qquad x \in \operatorname{supp}\rho \tag{34}$$

$$g'_+(x) - g'_-(x) = 0 \qquad x \notin \operatorname{supp}\rho \tag{35}$$

$$g'(z) = \frac{1}{z} + \mathcal{O}\left(\frac{1}{z^2}\right) \qquad z \to \infty. \tag{36}$$

We now have a system of equations that can be solved for a function $g'(z)$ analytic in $\mathbb{C} \setminus \mathbb{R}$.

Remember that in the convex V case the support is a compact interval $[\alpha, \beta]$, where α, β are still to be determined. In view of the system of equations for g', let's introduce the function

$$R(z) = (z - \alpha)^{\frac{1}{2}}(z - \beta)^{\frac{1}{2}}; \tag{37}$$

then, is it easy to see that

1. R is analytic on $\mathbb{C} \setminus [\alpha, \beta]$;
2. $R(z) = z + \mathcal{O}(1)$, as $z \to \infty$;
3. $R_+(x) = R_-(x)$ for $x \in \mathbb{R} \setminus [\alpha, \beta]$ and $R_+(x) = -R_-(x)$ for $x \in [\alpha, \beta]$.

Define now the function $h(z) := \frac{g'(z)}{R(z)}$. From property 2. and the asymptotic condition of g, we see that we must require that

$$h(z) = \frac{1}{z^2} + \mathcal{O}\left(\frac{1}{z^3}\right); \tag{38}$$

from property 3. and the equations for g,

$$h_+(x) - h_-(x) = \frac{V'(x)}{R_+(x)} \qquad x \in [\alpha, \beta]. \tag{39}$$

We can now use Sokhotski-Plemelj formula again to get

$$h(z) = \int_\alpha^\beta \frac{V'(x)}{R_+(x)(x-z)} \frac{\mathrm{d}x}{2\pi i} \qquad \Rightarrow \qquad g'(z) = R(z) \int_\alpha^\beta \frac{V'(x)}{R_+(x)(x-z)} \frac{\mathrm{d}x}{2\pi i}. \tag{40}$$

Finally, by expanding the integral expression of h for large z and imposing the behaviour (38), we have two equations to determine uniquely the endpoints α and β:

$$\int_\alpha^\beta \frac{V'(x)\,\mathrm{d}x}{R_+(x)} = 0 \qquad \int_\alpha^\beta \frac{xV'(x)\,\mathrm{d}x}{R_+(x)} = -2\pi i. \tag{41}$$

Paradigma (GUE). For $V(x) = \frac{x^2}{2}$ the OPs involved are the Hermite polynomials: the equilibrium density is

$$\rho(x) = \frac{1}{\pi}\sqrt{2 - x^2} \qquad x \in [-\sqrt{2}, \sqrt{2}]. \tag{42}$$

A quartic potential. For $V(x) = x^4$, the equilibrium density is

$$\rho(x) = \frac{1}{\pi}\left[2x^2 + \left(\frac{4}{3}\right)^{\frac{1}{2}}\right]\sqrt{\left(\frac{4}{3}\right)^{\frac{1}{2}} - x^2} \qquad x \in \left[-\left(\frac{4}{3}\right)^{\frac{1}{4}}, \left(\frac{4}{3}\right)^{\frac{1}{4}}\right]. \tag{43}$$

3.3 "Opening lenses"

Getting back to the RHP, let's perform one more transformation (the reason will be clear in the next subsection): we define

$$X(z) = e^{n\frac{\ell}{2}\sigma_2} W(z) e^{-n\frac{\ell}{2}\sigma_2}$$

then X solves the RHP with the same analyticity property and the same asympotic behaviour at infinity as W and with jumps

$$X_+(x) = X_-(x) \begin{bmatrix} e^{-n(g_+(x) - g_-(x))} & e^{-n(V(x) - g_+(x) - g_-(x) + \ell)} \\ 0 & e^{n(g_+(x) - g_-(x))} \end{bmatrix}, \qquad x \in \mathbb{R}. \tag{44}$$

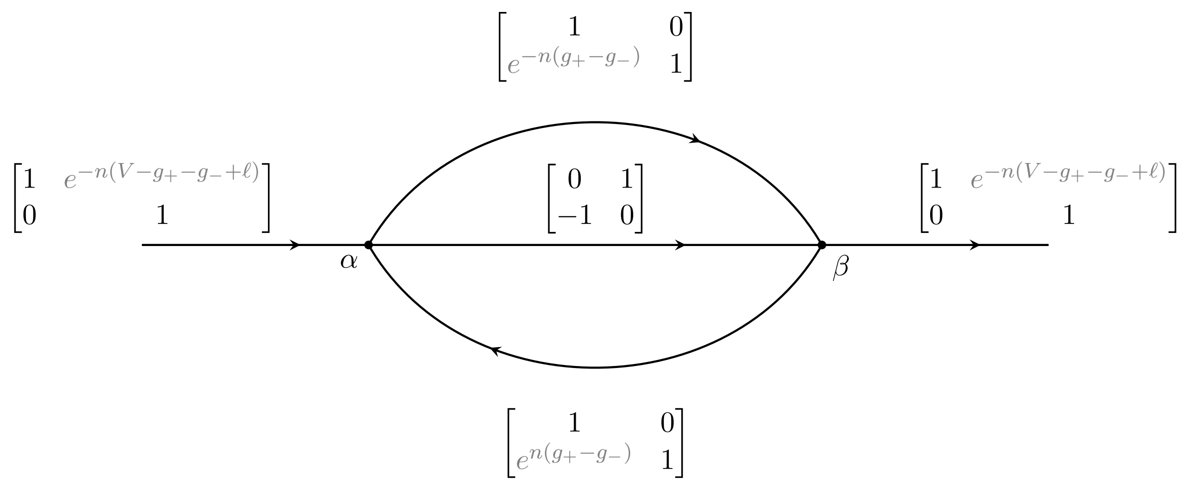

Figure 2: Riemann-Hilbert problem for Φ. Opening lenses: the entries in gray in the jump matrices are small in the regime as $n \to \infty$.

In particular, recall equations (31)–(32),

$$X_+(x) = X_-(x) \begin{bmatrix} 1 & e^{-n(V(x)-g_+(x)-g_-(x)+\ell)} \\ 0 & 1 \end{bmatrix} \qquad x \in \mathbb{R} \setminus [\alpha, \beta], \tag{45}$$

where the exponential term is asymptotically small when $n \to \infty$. Therefore, the jumps for X are close to the identity, at least in this part of the contour.

Also, from equations (27) and (31),

$$X_+(x) = X_-(x) \begin{bmatrix} e^{-n(g_+(x)-g_-(x))} & 1 \\ 0 & e^{n(g_+(x)-g_-(x))} \end{bmatrix} \qquad x \in [\alpha, \beta] \tag{46}$$

where $\theta(x) = 2\pi \int_x^\infty \rho(x)\,\mathrm{d}x$ is purely imaginary, since $\rho(x)$ is real; even more, because $\rho(x) > 0$ in the support, $\theta(x)$ is a real analytic function that is strictly decreasing.

According to the Cauchy–Riemann equations, $e^{-in\theta(z)}$ is exponentially small as $n \to \infty$ when z is fixed in the upper half-plane just above $[\alpha, \beta]$. Similarly $e^{in\theta(z)}$ is exponentially small as $n \to \infty$ when z is fixed in the lower half-plane just below $[\alpha, \beta]$.

We now have a (linear algebra) miracle!

Lemma 14.

$$\begin{bmatrix} e^{-n(g_+(x)-g_-(x))} & 1 \\ 0 & e^{n(g_+(x)-g_-(x))} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ e^{n(g_+(x)-g_-(x))} & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ e^{-n(g_+(x)-g_-(x))} & 1 \end{bmatrix} \tag{47}$$

The jump matrix on $[\alpha, \beta]$ splits into three jump matrices to which we can associate three contours (see Figure 2): an upper rim (or lens), a straight segment (i.e. the interval $[\alpha, \beta]$) and a lower rim. Indeed, we can analytic extend the matrices $\begin{bmatrix} 1 & 0 \\ e^{\mp n(g_+(z)-g_-(z))} & 1 \end{bmatrix}$ in the upper and lower half planes (respectively). This allows us to redefine X as

$$\Phi(z) = \begin{cases} X(z) & \text{outside the lenses} \\ X(z) \begin{bmatrix} 1 & 0 \\ -e^{-n(g_+(x)-g_-(x))} & 1 \end{bmatrix} & \text{in the upper lens } L_+ \\ X(z) \begin{bmatrix} 1 & 0 \\ e^{n(g_+(x)-g_-(x))} & 1 \end{bmatrix} & \text{in the lower lens } L_- \end{cases}$$

Then, Φ satisfies the RHP

Riemann–Hilbert problem 15. 1. $\Phi(z)$ is analytic $\forall\ z \in \mathbb{C} \setminus \{\mathbb{R} \cup L_+ \cup L_-\}$;

2. the boundary values are related by

$$\Phi_+(z) = \Phi_-(z) \begin{bmatrix} 1 & e^{-n(V(z)-g_+(z)-g_-(z)+\ell)} \\ 0 & 1 \end{bmatrix} \qquad x \in \mathbb{R},\ x < \alpha \text{ or } x > \beta \tag{48}$$

$$\Phi_+(z) = \Phi_-(z) \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad x \in [\alpha, \beta] \tag{49}$$

$$\Phi_+(z) = \Phi_-(z) \begin{bmatrix} 1 & 0 \\ e^{\mp n(g_+(x)-g_-(x))} & 1 \end{bmatrix} \qquad x \in L_\pm \tag{50}$$

3. $\Phi(z) = \mathbf{1} + \mathcal{O}\left(\frac{1}{z}\right)$ for $z \to \infty$.

Conclusion: the function $\Phi(z)$ has jumps that are asymptotically equal to the identity matrix everywhere except on the interval $[\alpha, \beta]$, where they are constant.

3.4 The model problem

Suppose that jumps outside the interval $[\alpha, \beta]$ and on the two lenses are actually equal the identity, not only asymptotically equal (meaning, there is no jump at all): then, we would be able to find an explicit solution!

We'll do it now and later we'll deal with showing that what we find is a quite decent approximation of Φ.

Define the matrix $\Psi(z)$ as a matrix-valued function that has the same jump as $\Phi(z)$ on the support of the measure and its same asymptotic at infinity:

$$\Psi_+(z) = \Psi_-(z) \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad z \in [\alpha, \beta] \tag{51}$$

$$\Psi(z) = \mathbf{1} + \mathcal{O}\left(\frac{1}{z}\right) \qquad z \to \infty \tag{52}$$

And also Ψ should have the "minimal" growth at $z = \alpha, \beta$ compatibly with the jump.

This RHP can be easily and explicitly solved!

We first notice that the jump matrix is constant and can be diagonalized:

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = F \cdot \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \cdot F^{-1}, \qquad \text{with } F = F^{-1} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -i \\ i & -1 \end{bmatrix}. \tag{53}$$

Thus, the matrix $\widehat{\Psi} = F^{-1}\Psi F$ has jumps

$$\widehat{\Psi}_+(z) = \widehat{\Psi}_-(z) \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \tag{54}$$

and same behaviour at infinity. This RHP decouples into two scalar RHPs, which can be solved easily using Sokhotski–Plemelji formula:

$$\widehat{\Psi}(z) = \begin{bmatrix} \left(\dfrac{z-\beta}{z-\alpha}\right)^{\frac{1}{4}} & 0 \\ 0 & \left(\dfrac{z-\beta}{z-\alpha}\right)^{-\frac{1}{4}} \end{bmatrix} = \left(\frac{z-\beta}{z-\alpha}\right)^{\frac{1}{4}\sigma_3} \tag{55}$$

and the solution $\Psi(z)$ is

$$\Psi(z) = \frac{1}{2} \begin{bmatrix} 1 & -i \\ i & -1 \end{bmatrix} \left(\frac{z-\beta}{z-\alpha}\right)^{\frac{1}{4}\sigma_3} \begin{bmatrix} 1 & -i \\ i & -1 \end{bmatrix}. \tag{56}$$

3.4.1 The error

We are now in a position of using the Small Norm Theorem: define the error matrix as

$$\mathcal{E}(z) = \Phi(z)\Psi^{-1}(z)$$

It solves a RHP with no jumps on $[a, b]$ (because Φ and Ψ have the same jump there) and on the rest of the contours (i.e. the rest of the real line and the upper/lower lenses) we have

$$\mathcal{E}_+(z) = \mathcal{E}_-(z)\Psi(z)\left[\mathbf{1} + \delta J(z)\right]\Psi^{-1}(z)$$

where we checked that $\|\delta J(z)\| \in \mathcal{O}\left(n^{-1}\right)$ for $n \to \infty$. Therefore,

$$\mathcal{E} = \mathbf{1} + \mathcal{O}\left(\frac{1}{n}\right) \qquad \text{in the limit as } n \to \infty. \tag{57}$$

Remark 16. *There is a not-negligible problem at the endpoints $z = \alpha, \beta$ where Ψ and Φ do not really match. This can be fixed by introducing a local approximation that fits well in the picture of the "model problem" with Ψ.*

One needs to add two fixed (small) disks around the endpoints. It can be shown that inside these disks the RHP can be solved exactly. The local solution is called the **local parametrix**. *In the generic case this can be constructed with the aid of Airy functions, but in non-generic situations (transitions of genus etc.) one needs special functions (Painlevé).*

We will not explain this part of the method in these notes.

3.5 Reaping the harvest and back to the kernel

Wrapping up, the chain of transformations we performed so far is:

$$Y \mapsto W \mapsto X = e^{-n\frac{\ell}{2}\sigma_3} Y e^{-n\left(g(z)-\frac{\ell}{2}\right)\sigma_3} \mapsto \Phi = \begin{cases} X\left[\mathbf{1} - e^{-n(g_+(x)-g_-(x))}\sigma_-\right] & \text{upper lens } L_+ \\ X & \text{outside} \\ X\left[\mathbf{1} + e^{n(g_+(x)-g_-(x))}\sigma_-\right] & \text{lower lens } L_- \end{cases}$$

Then we have argued (using the Small Norm Theorem) that

$$\Phi(z) \approx \Psi(z) = \frac{1}{2}\begin{bmatrix}1 & -i\\ i & -1\end{bmatrix}\left(\frac{z-\beta}{z-\alpha}\right)^{\frac{1}{4}\sigma_3}\begin{bmatrix}1 & -i\\ i & -1\end{bmatrix}, \tag{58}$$

Getting back to Y, we have that outside the lenses

$$Y(z) \approx e^{n\frac{\ell}{2}\sigma_3}\Psi(z)e^{n\left(g(z)-\frac{\ell}{2}\right)\sigma_3} \tag{59}$$

and inside the lenses (in particular, in the interior of the support of the equilibrium measure)

$$Y(z) \approx e^{n\frac{\ell}{2}\sigma_3}\Psi(z)\begin{bmatrix}1 & 0\\ \pm e^{\mp n(g_+(x)-g_-(x))} & 1\end{bmatrix}e^{n\left(g(z)-\frac{\ell}{2}\right)\sigma_3}. \tag{60}$$

Now, recall that

$$K_n(x,y) = \frac{e^{-\frac{n}{2}(V(x)+V(y))}}{2\pi i(x-y)}\left[Y_+^{-1}(y)Y_+(x)\right]_{21}; \tag{61}$$

writing out explicitly the product $Y_+^{-1}(y)Y_+(x)$ and considering the $(2,1)$-entry, we have:

$$\begin{aligned} e^{-\frac{n}{2}(V(x)+V(y))}\left[Y_+^{-1}(y)Y_+(x)\right]_{21} &\approx e^{n\left(-\frac{1}{2}V(y)+g_+(y)-\frac{\ell}{2}\right)}\left[e^{n(V(y)-2g_+(y)+\ell)} - e^{n(V(x)-2g_+(x)+\ell)}\right]e^{n\left(-\frac{1}{2}V(x)+g_+(x)-\frac{\ell}{2}\right)}\\ &= e^{-\frac{n}{2}\phi_+(y)}\left[e^{n\phi_+(y)} - e^{n\phi_+(x)}\right]e^{-\frac{n}{2}\phi_+(x)}\\ &= e^{-\frac{n}{2}(\phi_+(x)-\phi_+(y))} - e^{\frac{n}{2}(\phi_+(x)-\phi_+(y))}. \end{aligned} \tag{62}$$

where we called $\phi_+(x) = V(x) - 2g_+(x) + \ell$ and recalling (31). Now, using the fact that $\phi_+(z) = -g_+(z) + g_-(z) = -2\pi i\int_z^\infty \rho(t)\,\mathrm{d}t$:

$$e^{-\frac{n}{2}(\phi_+(x)-\phi_+(y))} - e^{\frac{n}{2}(\phi_+(x)-\phi_+(y))} = e^{i\pi n\int_x^y \rho(t)\mathrm{d}t} - e^{-i\pi n\int_x^y \rho(t)\mathrm{d}t} = 2i\sin\left(\pi n\int_x^y \rho(t)\mathrm{d}t\right) \tag{63}$$

Finally, we use the change of variables

$$x = x^* + \frac{\xi}{n\rho(x^*)} \qquad y = x^* + \frac{\eta}{n\rho(x^*)}$$

to conclude:

$$\frac{e^{-\frac{n}{2}(V(x)+V(y))}}{2\pi i(x-y)}\left[Y_+^{-1}(y)Y_+(x)\right]_{21} \approx \frac{2i}{2\pi i(\xi-\eta)}\sin\left(\pi n\int_{x^*+\frac{\xi}{n\rho(x^*)}}^{x^*+\frac{\eta}{n\rho(x^*)}}\rho(t)\mathrm{d}t\right) \approx \frac{\sin\left(\pi(\xi-\eta)\right)}{\pi(\xi-\eta)} \tag{64}$$

Et voilà:

$$\lim_{n\to+\infty}\frac{1}{n\rho(x^*)}K_n\left(x^*+\frac{\xi}{n\rho(x^*)}, x^*+\frac{\eta}{n\rho(x^*)}\right) = \frac{\sin\left(\pi(\xi-\eta)\right)}{\pi(\xi-\eta)}. \tag{65}$$

References

[1] P. Deift, T. Kriecherbauer, K.T.R. McLaughlin, S. Venakides, and X. Zhou. Uniform asymptotics for polynomials orthogonal with respect to varying exponential weights and applications to universality questions in random matrix theory. *Comm. Pure and Appl. Math*, 52:1335–1425, 1999.

[2] P. Deift and X. Zhou. A steepest descent method for oscillatory Riemann-Hilbert problems. *Bull. Amer. Math. Soc.*, 26(1):119–123, 1992.

[3] A. Fokas, A. Its, and A.V. Kitaev. The isomonodromy approach to matrix models in 2D quantum gravity. *Comm. Math. Phys.*, 197:395–430, 1992.

[4] E.B. Saff and V. Totik. *Logarithmic potentials with external fields*, volume 316 of *A series of comprehensive studies in Mathematics.* Springer, 1997.