Physical interpretation of the Phase Plane

Manuela Girotti

MATH 345 Differential Equations

Contents

We will analyze the dynamics of 1st-order differential equations with one space variable and 2nd-order conservative systems with one degree of freedom. These are the main cases where it is possible to obtain a complete description of the dynamics by inspecting the equations.

The notes are based on M. Girotti's personal notes from the course "Fisica Matematica I" given by Prof. Dario Bamusi at Università degli Studi di Milano in far 2007.

1 First-order autonomous differential equations

Consider a 1st-order ODE of the form

$$x' = f(x),$$

where the solution $x(t)$ can be interpreted as the description of the motion of a particle on the line $\mathbb{R}$.

The equation is clearly separable and in some cases it is possible to analytically calculate the solution (either in explicit or implicit) form. On the other hand, if we are interesting in the global dynamic of the set of solutions, we proceed as follows.

For simplicity, we will focus on one example:

$$x' = x^2 - 1$$

First of all, we notice that $x(t) = 1$ and $x(t) = -1$ are equilibrium points of the system (indeed, $x' = 0$). Consider now a coordinate system where the x-line is the line over which the motion of the particle takes place and the y-line represents it velocity. By drawing the graph of the function $y = f(x) = x^2 - 1$, we get a parabola, facing upwards and intersecting the x-axis on the two equilibrium points $x = \pm 1$.

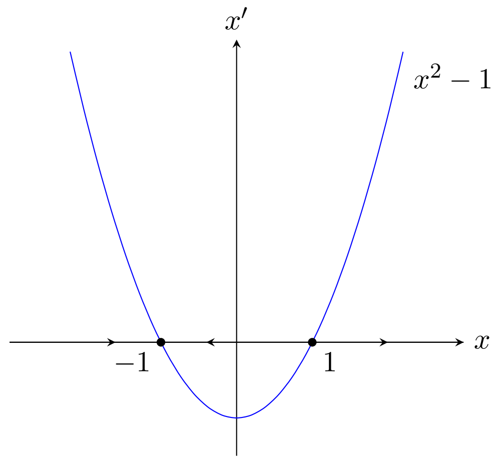
For any other point x_0 along the x-axis, we have that if the particles happens to be there (at x_0), then we can read its velocity from the value that the function f has in that point: $f(x_0)$. In particular, if $f(x_0) > 0$, then the velocity is positive, meaning that the particle will move rightwards, while if $f(x_0) < 0$, the velocity is negative, meaning that the particle will move rightwards.

In our example, we therefore have that for any initial condition x_0 that is bigger than 1 or smaller than -1, the velocity is positive, while for any initial condition $x_0 \in (-1, 1)$, the velocity is negative:

$$\forall\ x_0 \in (-\infty, -1) \cup (1, +\infty) :\ f(x_0) > 0 \ \Rightarrow\ \text{moving rightwards}$$
$$\forall\ x_0 \in (-1, 1) :\ f(x_0) < 0 \ \Rightarrow\ \text{moving leftwards}$$

From the graph we can easily see that $x = 1$ is unstable and $x = -1$ is (asymptotically) stable.

Note 1. We could have also checked the stability of these equilibrium points by checking their Jacobian (in this case is simply the derivative): $f'(x) = 2x$ and $f'(-1) = -2 < 0$ (asymptotically stable), while $f'(1) = 2 > 0$ (unstable).



2 Second-order conservative systems with one degree of freedom

Consider a mechanical system described by the following equation

$$x'' = F(x) \qquad \text{(Newton's law)},$$

where the solution $x(t)$ is the position of a particle (x'' represents its acceleration) and $F(x)$ represents the sum of all forces that are acting on the particle.

Note 2. It is important to notice that $F(x)$ does <u>not</u> depend on time: it is an **autonomous system** (in Physics it is called **isolated system**). Furthermore, $F(x)$ does <u>not</u> depend on x': it is a **conservative system** (as opposed to "dissipative" systems where the term x' appears and it introduces friction to the system).

Recall from basic Physics course that

$$F(x) = -\frac{\mathrm{d}V}{\mathrm{d}x}$$

where $V(x)$ is the **potential energy** of the system and the **total energy** of the system is the sum of the potential energy and the kinetic energy

$$E = \frac{1}{2}v^2 + V(x)$$

(where v represents the velocity: $v = x'$).

We want to prove that E is constant along the solutions of the ODE $x'' = F(x)$. Given a solution $x(t)$, consider the quantity

$$E(x(t), x'(t)) = \frac{1}{2}\left(x'\right)^2 + V(x)$$

and we want to prove that its time derivative is zero, using the fact that $x(t)$ is a solution of the equation above (this type of time derivative is called **Lie derivative** in the literature):

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\partial E}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial E}{\partial x'}\frac{\mathrm{d}x'}{\mathrm{d}t} = V'(x)x' + x'x'' = V'(x)x' + x'F(x) = V'(x)x' + x'(-V'(x)) = 0$$

Therefore, the energy E is indeed constant along the solutions of $x'' = F(x)$. We just proved what in Physics is called **Law of Conservation of Energy**: the total energy of an isolated system remains constant in time.

Definition 3. Any function $\Sigma = \Sigma(x, x')$ (where x is a solution of a differential equation) such that its Lie derivative is zero,

$$\frac{\mathrm{d}\Sigma}{\mathrm{d}t} = \frac{\partial \Sigma}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial \Sigma}{\partial x'}\frac{\mathrm{d}x'}{\mathrm{d}t} = 0,$$

is called **constant of motion** of the system.

Application. Given initial conditions $x(0) = x_0$ and $x'(0) = v_0$, we calculate the initial energy

$$E_0 = \frac{1}{2}v_0^2 + V(x_0)$$

this will be equal to some quantity E_0 which will then remain constant in time. The solution $(x(t), x'(t))$ moves only along the orbit (in the (x, x')-phase plane) that is identified by the equation

$$\frac{1}{2}(x')^2 + V(x) = E_0.$$

The equation above is indeed the equation for the orbits of the system.

Example – The Harmonic Oscillator. As a first example, we can consider the Harmonic Oscillator (with frequency $\omega = 1$):

$$x'' = -x$$

This gives a total energy equal to:

$$F(x) = -x \quad \Leftrightarrow \quad V(x) = -\int F(x)\,\mathrm{d}x = \frac{1}{2}x^2$$

$$E(x, x') = \frac{1}{2}\left(x'\right)^2 + \frac{1}{2}x^2$$

Therefore, for any fixed quantity E_0, the equation $(x')^2 + x^2 = 2E_0$ describes a curve which happens to be an orbit of the system in the phase plane: it is a circle with radius $\sqrt{2E_0}$.

Note 4. In the general case of the harmonic oscillator, $x'' = -\omega^2 x$, the orbits would have been equal to ellipses.

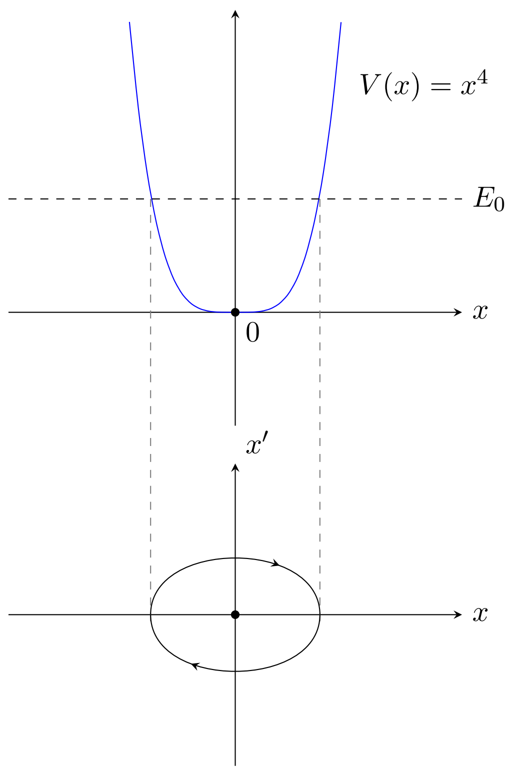
Example – General potential In general, $V(x)$ could have a more complicated expression, but we can still study the orbits of a system in an effective way by studying the total energy $E = \frac{1}{2}(x')^2 + V(x)$. First of all, we notice that we can solve for x' (once we fixed a value for the total energy):

$$x' = \pm\sqrt{2\left[E_0 - V(x)\right]}$$

Clearly, this expression makes sense only if $E_0 - V(x) \geq 0$, which identifies the region in space that is accessible by the particle when it has energy equal to E_0. For further analysis, we consider again one simple example:

$$V(x) = x^4 \qquad \text{meaning } x'' = -4x^3.$$

First of all, by plotting the graph of $V(x)$ we can already see that if $E_0 < 0$, then no dynamic is happening ($E_0 - V(x) < 0$). If $E_0 = 0$, then the only accessible region is given by the point $x = 0$ (which is the equilibrium point of the system). For $E_0 > 0$ we have the following. The



equation $V(x) = E_0$ gives two solutions $\pm\sqrt[4]{E_0}$; therefore, the accessible region for the particle is $x \in [-\sqrt[4]{E_0}, \sqrt[4]{E_0}]$.

Assume that our initial conditions for the system are $x(0) = x_0 = -\sqrt[4]{E_0}$ and $x'(0) = v_0 = 0$ (point $(-\sqrt[4]{E_0}, 0)$ on the phase plane): the initial velocity is zero, but the acceleration is positive ($x'' = -x_0^3 > 0$), therefore oriented rightward. This implies that the particle will start moving rightward and x increases.

On the other hand, as x increases and the particle moves, the potential energy decreases and the velocity becomes non negative (the kinetic energy increases), always according to the law $\frac{1}{2}(x')^2 + V(x) = E_0$.

This continues till $x = 0$: this is the case where the potential energy is zero and we have maximum kinetic energy (maximum velocity). Then the particles continues as x increases towards the other endpoint $+\sqrt[4]{E_0}$ and progressively gains again potential energy and loses velocity. Once the point $(+\sqrt[4]{E_0}, 0)$ is hit by the particle, the particle moves now backwards following the same trajectory, but with opposite velocity (this is the lower branch of the orbit).

If we pick another value of E_0, we notice that nothing changes: the orbits look all the same (either bigger or smaller than the previous one) and they are all closed curves, meaning that any solution is periodic.

Note 5. Notice that the dynamics of such conservative systems is completely determined by the expression of the potential energy only.

Exercise. Study the phase space and stability of the system given by the following equation:

$$x'' = 2x - 4x^3.$$

Hint: the potential is $V(x) = -\int 2x - 4x^3 \,\mathrm{d}x = x^4 - x^2 = x^2(x^2 - 1)$.

2.1 The period of an oscillation

Given a conservative system with differential equation of the type

$$x'' = -V'(x),$$

we saw that the orbits in the phase plane are described by the energy equation

$$\frac{1}{2}v^2 + V(x) = E_0$$

where x is the position and $v = x'$ is the velocity; the constant E_0 is the energy of the system and it is identified by the initial condition of the corresponding Cauchy problem.

In the case when the potential $V(x)$ gives rise to periodic solution (closed orbits in the phase plane), we can additionally use this formula to calculate the period T of these oscillatory solutions, meaning we can calculate the time needed to go around the closed orbit once.

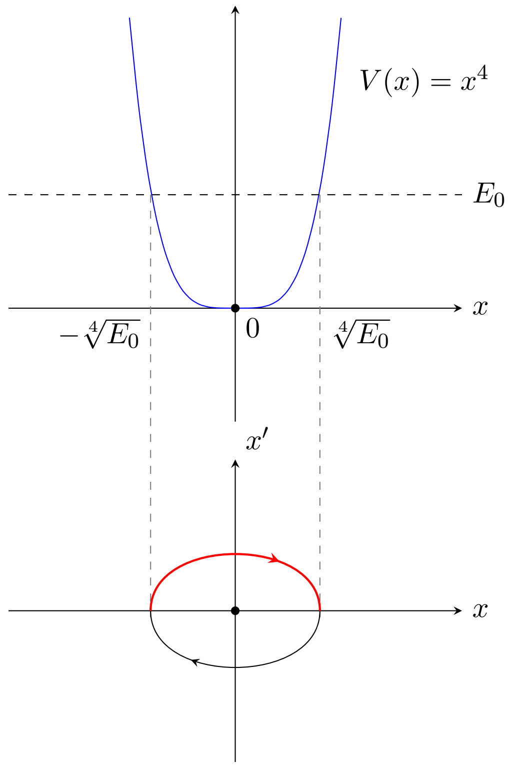
Solving for $v = x'$ we have

$$x' = \pm\sqrt{2\left[E_0 - V(x)\right]}$$

and we can easily recognize this equation to be a first order separable equation that can be analytically solved:

$$\int_{x_0}^{x(t)} \frac{\mathrm{d}x}{\sqrt{2\left[E_0 - V(x)\right]}} = \int_{t_0}^{t} \mathrm{d}s = t - t_0.$$

First of all, we divide the orbit into two sections: half of of it lies in the upper half-plane, where the velocity is positive ($x' > 0$), and the other half lies in the lower half-plane, where the velocity



is negative ($v < 0$). It is easy to see that we can just calculate the integral over the first half of the orbit (the positive one), because the second half is analogous. This is why we only considered the $+$ side of the square root (see the red curve in the picture).

For simplicity, let's consider again a potential of the form $V(x) = x^4$ as in the example above. We fix an initial energy $E_0 > 0$ such that our initial condition will be $x(0) = x_0 = -\sqrt[4]{E_0}$ (with zero velocity, $x'(0) = 0$) and the final position will be at $x(T) = \sqrt[4]{E_0}$ (with zero velocity as well, $x'(T) = 0$).

The period of this oscillation will be equal to

$$T = 2\int_{x_0}^{x(T)} \frac{\mathrm{d}x}{\sqrt{2\,(E_0 - x^4)}} = 2\int_{-\sqrt[4]{E_0}}^{\sqrt[4]{E_0}} \frac{\mathrm{d}x}{\sqrt{2\,(E_0 - x^4)}}$$

Remark 6. *Notice that the integral is an improper integral, since the square root in the denominator vanishes at the end points $\pm\sqrt[4]{E_0}$, but the integral is convergent!*

Usually, integrating such integrals by hand can be performed only in simple cases, but with the help of a computer it can be easily handled. In this case, we can further simplify the expression:

$$T = 4\int_0^{\sqrt[4]{E_0}} \frac{\mathrm{d}x}{\sqrt{2\,(E_0 - x^4)}} = \int_0^1 \frac{4\sqrt[4]{E_0}\,\mathrm{d}u}{\sqrt{E_0}\sqrt{2(1-u^4)}} = \frac{1}{\sqrt[4]{E_0}}\int_0^1 \frac{4\,\mathrm{d}u}{\sqrt{2(1-u^4)}}$$

in the first equality we used the fact that x^4 is an even function over an even domain and we performed a change of variables $u = \frac{x}{\sqrt[4]{E_0}}$. The integral is an elliptic integral and it can be integrated with the use of advanced integral tables or with the help of a computer.

For our purposes, we are not really interested in the exact value of it, but we can say that it will be equal to some positive number κ (which does not depend on the energy):

$$T = \frac{\kappa}{\sqrt[4]{E_0}}$$

We can now observe that the highest the energy of the system, the smaller the time to complete the orbit. Also, the period T tends to infinity when the energy E_0 tends to zero. This result might seem counterintuitive, but it can be explained by noticing that with high energy, the force that it's applied to the particle is very high as well, therefore the particle travels faster.

An analogue phenomenon is the so called **gravitational slingshot**: in order to send a satellite in orbit, it is not so convenient to send it directly towards its target, but it is better to send it towards regions of the space (close to planets) where the gravitational forces are very strong such that they impart a big acceleration to the satellite that will travel faster towards the destination, saving both time and fuel!

Exercise. Calculate the period of the oscillations for the harmonic oscillator

$$x'' = -x.$$

Will the period depend on the energy?