Additional Notes on Power Series

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MATH 317-01 Advanced Calculus of one variable

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1 Quick recall

We recall here the main facts about power series. A detailed description of the topic has been covered a few weeks ago and it refers to the Section 2.6.5 of the textbook.

Definition 1. Let $\{a_n\}$ be a sequence of real numbers and $c \in \mathbb{R}$. The **power series** centered at c with coefficients $\{a_n\}$ is the function

$$\sum_{n=0}^{+\infty} a_n (x-c)^n.$$

Every power series is a function defined on a certain domain (called **interval of convergence**) that is identified by the **radius of convergence** R:

Theorem 2. *Let*

$$f(x) := \sum_{n=0}^{+\infty} a_n (x-c)^n$$

be a power series. There exists a constant $R \in \mathbb{R} \cup \{+\infty\}$ such that the series converges absolutely for $|x-c| < R$ (i.e. for any $x \in (c-R, c+R)$) and diverges for $|x-c| > R$.

Theorem 3 (Hadamard's Theorem). *The radius of convergence R of the power series $f(x) := \sum_{n=0}^{+\infty} a_n (x-c)^n$ is given by*

$$R = \frac{1}{\limsup_{n\to+\infty} \sqrt[n]{|a_n|}}.$$

Remark 4. *The radius of convergence of a power series can also be calculated as*

$$R = \frac{1}{\lim_{n\to+\infty} \left|\frac{a_{n+1}}{a_n}\right|}.$$

2 Abel's Theorem

From the theorems above we have that given a power series, the series converges absolutely for any point x in the open interval $(c-R, c+R)$, however no information is given about what happens at the endpoints $x = c \pm R$ and in general the series may converges or diverges there.

On the other end, suppose that we have a power series (say, centered at $c = 0$)

$$f(x) := \sum_{n=0}^{+\infty} a_n x^n$$

with radius of convergence R (therefore the series converges for any $x \in (-R, R)$) and suppose that for example the series $\sum_{n=0}^{\infty} a_n R^n < +\infty$. How do we know that

$$f(R) = \sum_{n=0}^{\infty} a_n R^n?$$

(i.e. we just plug the value R into the definition of the power series)

To state the problem in a more correct way, we need to prove that assuming that $\sum_{n=0}^{\infty} a_n R^n < +\infty$, then

$$\lim_{x \to R_-} f(x) = \sum_{n=0}^{\infty} a_n R^n.$$

The following theorem has a positive answers to this problem. We are considering here the simple(r) case where the center of the series is $c = 0$, the radius of convergence is $R = 1$ and we'll be checking only the right end point of the interval of convergence, namely $x = 1$.

The result can be easily generalized to any power series centered at a point $c \in \mathbb{R}$ with radius of convergence $R > 0$ and with x equal either one of the endpoints of the interval of convergence.

Theorem 5 (Abel's Theorem). *Let $f(x) := \sum_{n=0}^{+\infty} a_n x^n$ a power series with radius of convergence $R = 1$. If $\sum_{n=0}^{\infty} a_n < +\infty$, then*

$$\lim_{x \to 1_-} f(x) = \sum_{n=0}^{\infty} a_n.$$

Proof. The main tool used here will be the **summation by parts formula** for two finite sequences $\{u_1, \ldots, u_n\}$ and $\{v_0, \ldots, v_n\}$:

$$\sum_{k=1}^{n} u_k(v_k - v_{k-1}) = u_n v_n - u_1 v_0 - \sum_{k=1}^{n-1} v_k(u_{k+1} - u_k)$$

1. We know that $\sum_{n=0}^{\infty} a_n < +\infty$, meaning that the sequence of partial sums $\{s_n := \sum_{k=0}^{n} a_n\}$ converges ($s_n \to S$). We recall also that $a_n = s_n - s_{n-1}$ for all $n \in \mathbb{N}$ and $a_0 = s_0$.

   Now, we will use the summation by parts formula for $u_k = x^k$ and $v_k = s_k$

$$\sum_{k=0}^{n} a_k x^k = a_0 + \sum_{k=1}^{n} (s_k - s_{k-1}) x^k = a_0 + s_n x^n - s_0 x - \sum_{k=1}^{n-1} s_k (x^{k+1} - x^k)$$

$$= a_0(1-x) + s_n x^n + \sum_{k=1}^{n-1} s_k (1-x) x^k = s_n x^n + \sum_{k=0}^{n-1} s_k (1-x) x^k$$

This chain of equality is always valid and additionally we know that for any $|x| < 1$ the left hand side of the identity is (absolutely) convergent. Also, the term $s_n x^n \to 0$ as $n \to +\infty$, because $x^n \to 0$ ($|x| < 1$) and the sequence of partial sums $\{s_n\}$ of the (numeric series) $\sum_n a_n$ is convergent, therefore it is bounded.

Taking the limit as $k \to +\infty$ on both sides, we have

$$\sum_{k=0}^{\infty} a_k x^k = (1-x) \sum_{k=0}^{\infty} s_k x^k.$$

2. If $S = \sum_{n=0}^{\infty} a_n = \lim_{n\to+\infty} s_n$ is the sum of the (numeric) series, we want to show that $\sum_{n=0}^{\infty} a_n x^n \to S$ as $x \nearrow 1_-$: first of all

$$\sum_{n=0}^{\infty} a_n x^n - S = (1-x) \sum_{n=0}^{\infty} s_n x^n - S = (1-x) \sum_{n=0}^{\infty} (s_n - S) x^n$$

(the second equality comes from the fact that $(1-x)\sum_{n=0}^{\infty} x^n = (1-x)\frac{1}{1-x} = 1$, geometric series).

3. Then, since $s_n \to S$, this means that $\forall\, \epsilon > 0\ \exists\, M_\epsilon \in \mathbb{N}$ such that $|s_n - S| < \epsilon$ for $n \geq M_\epsilon$; so, we fix $\epsilon > 0$ and we split the series (the first term will be a just a finite sum, therefore bounded)

$$\left|(1-x)\sum_{n=0}^{\infty}(s_n - S)x^n\right| \leq (1-x)\sum_{n=0}^{M_\epsilon - 1} |s_n - S|\,|x|^n + (1-x)\sum_{n=M_\epsilon}^{\infty} |s_n - S|\,|x|^n \leq$$
$$\leq (1-x)K + (1-x)\epsilon \sum_{n=0}^{\infty} |x|^n = (1-x)K + (1-x)\epsilon \frac{1}{1-|x|}$$

where we don't write $|1-x|$ because $|x| < 1$; also, since we want to take the limit as $x \nearrow 1_-$, we can assume that $0 < x < 1$. In conclusion,

$$\left|(1-x)\sum_{n=0}^{\infty}(s_n - S)x^n\right| \leq K(1-x) + \epsilon \to 0.$$

□

Example 1. The series

$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n$$

has radius of convergence

$$R = \frac{1}{\limsup_{n\to+\infty} \sqrt[n]{\left|\frac{(-1)^{n+1}}{n}\right|}} = \frac{1}{\limsup_{n\to+\infty} n^{-\frac{1}{n}}} = 1$$

so it converges for $|x-1|<1$ and it diverges for $|x-1|>1$. We still need to check the endpoints, i.e. $x=0$ and $x=2$.

For $x=2$ the power series becomes an alternating harmonic series $f(2)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$ which converges thanks to Leibniz's rule. For $x=0$, the series become

$$f(0)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}(-1)^n=-\sum_{n=1}^{\infty}\frac{(-1)^{2n}}{n}=-\sum_{n=1}^{\infty}\frac{1}{n}$$

which is a harmonic series and therefore diverges.

In conclusion, the interval of convergence is $(0,2]$.

Example 2. The series

$$f(x)=\sum_{n=0}^{\infty}(-1)^n x^{2^n}=x-x^2+x^4-x^8+x^{16}-x^{32}+\dots$$

has coefficients

$$a_n=\begin{cases}1 & \text{if } n=2^k\\ 0 & \text{if } n\neq 2^k\end{cases}$$

and its radius of convergence is equal to

$$R=\frac{1}{\limsup_{n\to+\infty}\sqrt[n]{|a_n|}}=1$$

so it converges for $|x|<1$ and it diverges for $|x|>1$.

At the endpoints we have that $f(1)=f(-1)=\sum_{n=0}^{\infty}(-1)^n$ which is clearly a divergent series. Therefore the interval of convergence is $(-1,1)$.

Note 6. *This series is called* **lacunary power series** *because there are successively longer gaps ("lacuna") between the powers with non-zero coefficients.*

3 Differentiation and Integration of Power series

We will now tackle the problem of differentiating and integrating power series.

Theorem 7 (**Term-by-term differentiation of a power series**)**.** *Suppose that the power series*

$$f(x)=\sum_{n=0}^{\infty}a_n(x-c)^n$$

has radius of convergence R. Then the power series

$$g(x)=\sum_{n=1}^{\infty}na_n(x-c)^{n-1}$$

is well defined and it also has radius of convergence R.

Moreover, the power series f is differentiable $\forall\, x\in(c-R,c+R)$ and we have the identity

$$f'(x)=g(x)=\sum_{n=1}^{\infty}na_n(x-c)^{n-1}.$$

We will see only the proof of the first part of the theorem here. To prove that the power series g is indeed the derivative of the power series f some extra tools from Analysis are required, that we don't possess at the moment.

Proof. Assume without loss of generality that $c = 0$. Let $|x| < R$: this implies (least upper bound properties of the set $\mathbb{R}$) that there exists $\rho > 0$ such that $|x| < \rho < R$ and let

$$r := \frac{|x|}{\rho} \in (0, 1).$$

To estimate the terms in the differentiated power series by the terms in the original series, we write

$$|na_n x^{n-1}| = \frac{n}{\rho}\left(\frac{|x|}{\rho}\right)^{n-1}|a_n|\rho^n = \frac{nr^{n-1}}{\rho}|a_n|\rho^n$$

The series $\sum_{n=1}^{\infty} nr^{n-1}$ converges by ratio test, since

$$\lim_{n\to\infty} \frac{(n+1)r^n}{nr^{n-1}} = \lim_{n\to\infty} r\left(1 + \frac{1}{n}\right) = r < 1$$

therefore the sequence $\{nr^{n-1}\}$ is bounded (it actually goes to zero): $-M \le nr^{n-1} < M$, $\forall n \in \mathbb{N}$. Therefore,

$$\left|na_n x^{n-1}\right| = \frac{nr^{n-1}}{\rho}|a_n|\rho^n \le \frac{M}{\rho}|a_n|\rho^n \qquad \forall n \in \mathbb{N}$$

and the series $\sum |a_n|\rho^n$ converges since $\rho < R$ (remember that for all $|x| < R$ the power series converges absolutely).

Finally, the comparison test implies that $\sum na_n x^{n-1}$ converges absolutely.

Conversely, suppose that $|x| > R$, then $\sum |a_n|x^n$ diverges (since $\sum a_n x^n$ diverges) and

$$\left|na_n x^{n-1}\right| \ge \frac{1}{|x|}\left|a_n x^n\right| \qquad \forall n \in \mathbb{N},$$

so the comparison test implies that $\sum na_n x^{n-1}$ diverges.

In conclusion, the power series $g(x) = \sum_{n=1}^{\infty} na_n(x-c)^{n-1}$ has the same radius of convergence R as the power series f. □

Similarly, using the theorem above and the fundamental theorem of calculus, the following theorem for integration of power series can be proven.

Theorem 8 (Term-by-term integration of a power series). *Suppose that the power series*

$$f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$$

has radius of convergence R. Then the power series

$$g(x) = \sum_{n=0}^{\infty} a_n \frac{(x-c)^{n+1}}{n+1} + C$$

is well defined and it also has radius of convergence R.

Moreover, the power series g is one anti-derivative of the power series f.

Example 1. Find a power series representation for the function (wherever it makes sense)

$$f(x) = \frac{1}{(1-x)^2}.$$

We have that

$$\frac{1}{(1-x)^2} = \left[\frac{1}{1-x}\right]'$$

and for $|x| < 1$ (geometric series; its radius of convergence is $R = 1$)

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.$$

Therefore, for $|x| < 1$ (the radius of convergence is also equal 1)

$$\frac{1}{(1-x)^2} = \left[\sum_{n=0}^{\infty} x^n\right]' = \sum_{n=0}^{\infty} n x^{n-1} = \sum_{n=0}^{\infty} (n+1)x^n.$$

Example 2. Find a power series representation for the function (wherever it makes sense)

$$f(x) = \arctan(x).$$

We know that

$$\arctan(x) = \int \frac{\mathrm{d}x}{1+x^2}$$

and for $|x| < 1$ (geometric series; its radius of convergence is $R = 1$)

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-1)^n x^{2n}.$$

Therefore, for $|x| < 1$ (the radius of convergence is also equal 1)

$$\arctan(x) = \int \sum_{n=0}^{\infty} (-1)^n x^{2n} = C + \sum_{n=0}^{\infty} (-1)^n \int x^{2n} \mathrm{d}x = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

and since $\arctan(0) = 0$, then $C = 0$. In conclusion,

$$\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

with radius of convergence equal 1.