Notes on Riemann Integral

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MATH 317-01 Advanced Calculus of one variable

These notes will explain the classical theory of integration due to B. Riemann. Throughout the notes we will always assume that

a) the function f is defined on a closed bounded interval $f : [a, b] \to \mathbb{R}$

b) the function f is bounded: $m \leq f(x) \leq M$ for all $x \in [a, b]$.

Note 1. *It's important to set the distinction between the (Riemann) integral and the antiderivative. The Riemann integral is the "area" under the graph of a function. The antiderivative is the "reverse" of the derivative.*

The link between these two concepts is given by the Fundamental Theorem of Calculus that will be explained and proved within these notes.

1 Definition and first properties

Definition 2. Consider a closed bounded interval $[a, b]$. A **partition** P of $[a, b]$ is a (finite) set of numbers

$$a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b.$$

Let $f : [a, b] \to \mathbb{R}$ be a bounded function and P be a partition of $[a, b]$, then we can define

$$m_i := \inf \{ f(x) \mid x \in [x_{i-1}, x_i] \}$$
$$M_i := \sup \{ f(x) \mid x \in [x_{i-1}, x_i] \}$$

Definition 3. We call **lower sum** of f with respect to the partition P the quantity

$$L(f, P) := m_1(x_1 - x_0) + m_2(x_2 - x_1) + \ldots + m_n(x_n - x_{n-1}) = \sum_{i=1}^{n} m_i \Delta x_i$$

where $\Delta x_i := x_i - x_{i-1}$.

Analogously, we define the **upper sum** of f with respect to the partition P the quantity

$$U(f, P) := M_1(x_1 - x_0) + M_2(x_2 - x_1) + \ldots + M_n(x_n - x_{n-1}) = \sum_{i=1}^{n} M_i \Delta x_i$$

Remark 4 (Geometric interpretation). *If the function f is non negative ($f(x) \geq 0$ for all $x \in [a, b]$), then for any partition P of the interval $[a, b]$, the lower sum $L(f, P)$ is equal to the sum of the areas of some rectangles which have base equal to Δx_i and height equal to m_i. Similarly, the upper sum $U(f, P)$ is the sum of the areas of rectangles whose base is equal to Δx_i and height is equal to M_i.*

Proposition 5. *Given a function $f : [a, b] \to \mathbb{R}$ bounded. In particular, let*

$$m := \inf_{[a,b]} f(x) \qquad M := \sup_{[a,b]} f(x)$$

Then, for any partition P of $[a, b]$, we have

$$m(b-a) \leq L(f, P) \leq U(f, P) \leq M(b-a)$$

Proof. It follows from the fact that $m \leq m_i \leq M_i \leq M$ for all $i = 1, \ldots, n$.

$$m(b-a) = m\sum_{i=1}^{n} \Delta x_i = \sum_{i=1}^{n} m\Delta x_i \leq \sum_{i=1}^{n} m_i\Delta x_i \,(= L(f,P)) \leq \sum_{i=1}^{n} M_i\Delta x_i \,(= U(f,P)) \leq$$
$$\leq \sum_{i=1}^{n} M\Delta x_i = M\sum_{i=1}^{n} \Delta x_i = M(b-a).$$

□

<u>*Consequence:*</u> the sets

$$\mathcal{L} := \{L(f,P) \,|\, P = \text{ partition of } [a,b]\}$$
$$\mathcal{U} := \{U(f,P) \,|\, P = \text{ partition of } [a,b]\}$$

are bounded sets in $\mathbb{R}$, therefore there exist $\sup \mathcal{L}$ and $\inf \mathcal{U}$.

Definition 6. Let $f : [a, b] \to \mathbb{R}$ be a bounded function, then f is **Riemann integrable** (and we denote it as $f \in \mathcal{R}([a, b])$) if

$$\sup_P L(f,P) = \inf_P U(f,P).$$

We call the **(Riemann) integral** of f over $[a, b]$ the real number

$$\sup_P L(f,P) = \inf_P U(f,P) =: \int_a^b f(x)\mathrm{d}x$$

Remark 7 (Geometric interpretation). *If the function f is non negative ($f(x) \geq 0$ for all $x \in [a, b]$), we can define*

$$T = \left\{(x,y) \in \mathbb{R}^2 \,|a \leq x \leq b,\ 0 \leq y \leq f(x)\right\}.$$

Then for any partition P of the interval $[a, b]$, the lower sum $L(f, P)$ is the area of a multi-rectangle that is contained in T and the upper sum $U(f, P)$ is the area of a multi-rectangle that contains T.

If $f \in \mathcal{R}([a, b])$, then

$$\text{area}(T) = \sup_P L(f,P) = \inf_P U(f,P) = \int_a^b f(x)\mathrm{d}x$$

Proposition 8. *Let $f : [a, b] \to \mathbb{R}$ a bounded function.*

a) for any choice of partitions P_1 and P_2, $L(f, P_1) \le U(f, P_2)$

b) $f \in \mathcal{R}([a, b])$ if and only if $\forall\ \epsilon > 0\ \exists\ P^\star$ partition of $[a, b]$ such that

$$U(f, P^\star) - L(f, P^\star) < \epsilon.$$

Proof. a) Suppose we have a partition $P = \{x_0 < x_1 < \ldots < x_{n-1} < x_n\}$ and we add one point y to it: we call $\widetilde{P} = \{x_0 < x_1 < \ldots < x_{j-1} < y < x_j < \ldots < x_{n-1} < x_n\}$ the new partition.

Then, we have that m_i and M_i are always the same for all the intervals $[x_{i-1}, x_i]$, except the j-th one where we have

$$m_j = \inf_{[x_j - x_{j-1}]} f(x) \le \begin{cases} \widetilde{m_j} := \inf_{[x_{j-1}, y]} f(x) \\ \widetilde{\widetilde{m_i}} := \inf_{[y, x_j]} f(x) \end{cases}$$
$$M_j = \inf_{[x_j - x_{j-1}]} f(x) \ge \begin{cases} \widetilde{M_j} := \sup_{[x_{j-1}, y]} f(x) \\ \widetilde{\widetilde{M_i}} := \sup_{[y, x_j]} f(x) \end{cases}$$

Calculating the lower sum, we have that (split the interval $[x_{j-1}, x_j]$ into two intervals $[x_{j-1}, y]$ and $[y, x_j]$)

$$\begin{gathered} L(f, P) = m_1 \Delta x_1 + \ldots + m_j \Delta x_j + \ldots + m_n \Delta x_n = \\ m_1 \Delta x_1 + \ldots + m_j (x_j - x_{j-1} + y - y) + \ldots + m_n \Delta x_n \le \\ \le m_1 \Delta x_1 + \ldots + \widetilde{m_j}(y - x_{j-1}) + \widetilde{\widetilde{m_i}}(x_j - y) + \ldots + m_n \Delta x_n = L(f, \widetilde{P}); \end{gathered}$$

the same holds for the upper sum:

$$\begin{gathered} U(f, P) = M_1 \Delta x_1 + \ldots + M_j \Delta x_j + \ldots + M_n \Delta x_n \ge \\ \ge M_1 \Delta x_1 + \ldots + \widetilde{M_j}(y - x_{j-1}) + \widetilde{\widetilde{M_i}}(x_j - y) + \ldots + M_n \Delta x_n = U(f, \widetilde{P}). \end{gathered}$$

We can repeat this argument by adding a finite number of points.

Therefore, given two partitions P_1 and P_2, let's consider the partition $\widetilde{P} = P_1 \cup P_2$ (the partition considering all the points of P_1 and P_2) and we have

$$L(f, P_1) \le L(f, \widetilde{P}) \le U(f, \widetilde{P}) \le U(f, P_2)$$

b)

($\Rightarrow$) If $f \in \mathcal{R}([a, b])$, then we have $\sup_P L(f, P) = \inf_P U(f, P)$. By the properties of sup and inf, for any $\epsilon > 0$ there exist two partitions P_1 and P_2 such that

$$L(f, P_1) < \sup_P L(f, P) - \frac{\epsilon}{2} \qquad \text{and} \qquad U(f, P_2) > \inf_P U(f, P) + \frac{\epsilon}{2}$$

Take the partition $P^\star := P_1 \cup P_2$, then thanks to point a) above

$$U(f, P^\star) - L(f, P^\star) \le U(f, P_2) - L(f, P_1) < \cancel{\inf_P U(f, P)} + \frac{\epsilon}{2} - \cancel{\sup_P L(f, P)} + \frac{\epsilon}{2} = \epsilon.$$

($\Leftarrow$) Viceversa, if for any $\epsilon > 0$ there exists a partition $P^\star$ such that $U(f, P^\star) - L(f, P^\star) < \epsilon$, since $L(f, P^\star) \leq \sup_P L(f, P) \leq \inf_P U(f, P) \leq U(f, P^\star)$, it follows that

$$\inf_P U(f, P) - \sup_P L(f, P) < U(f, P^\star) - L(f, P^\star) < \epsilon \longrightarrow 0.$$

Therefore, $\inf_P U(f, P) = \sup_P L(f, P)$ and $f \in \mathcal{R}([a, b])$. □

Definition 9. Given a partition P, we call a **refinement** of P a new partition $\widetilde{P}$ that has the same points as P plus some extra (finite number of) points.

Remark 10. *It follows from the previous proposition that in general we have*

$$\sup_P L(f, P) \leq \inf_P U(f, P)$$

and the equality is achieved only for integrable functions $f \in \mathcal{R}([a, b])$.

Example 1. Let $f(x) = c \; \forall \; x \in [a, b]$ a constant function. For any partition P of $[a, b]$ we have that $m_i = M_i = c$ for all $i = 1, \ldots, n$. Therefore, $\forall \; P$

$$L(f, P) = U(f, P) = \sum_{i=1}^{n} c\Delta x_i = c(b - a)$$

Taking the sup and inf we still get the same number, therefore $f \in \mathcal{R}([a, b])$ and $\int_a^b f(x)\mathrm{d}x = \sup_P L(f, P) = \inf_P U(f, P) = c(b - a)$.

Example 2. Consider the Dirichet's function over the interval $[0, 1]$

$$f(x) = \begin{cases} 1 & x \in [0, 1] \cap \mathbb{Q} \\ 0 & x \in [0, 1] \cap \mathbb{R} \backslash \mathbb{Q} \end{cases}$$

For any partition P we have that $m_i = 0$ and $M_i = 1$ for all $i = 1, \ldots, n$, therefore $L(f, P) = 0$ and $U(f, P) = 1$. Taking the sup and inf we still get the same values:

$$\sup_P L(f, P) = 0 < \inf_P U(f, P) = 1$$

therefore $f \notin \mathcal{R}([a, b])$.

Example 3. Let $f : [0, 2] \to \mathbb{R}$

$$f(x) = \begin{cases} 1 & x \in [0, 1) \\ \frac{1}{2} & x = 1 \\ 0 & x \in (1, 2] \end{cases}$$

Let $0 < \epsilon < 1$ and consider the partition $\overline{P} = \{0 < x_1 := 1 - \epsilon < x_2 := 1 + \epsilon < 2\}$, then we have

$$m_1 = \inf_{[0,1-\epsilon]} f(x) = 1 \qquad M_1 = \sup_{[0,1-\epsilon]} f(x) = 1$$
$$m_2 = \inf_{[1-\epsilon,1+\epsilon]} f(x) = 0 \qquad M_2 = \sup_{[1-\epsilon,1+\epsilon]} f(x) = 1$$
$$m_3 = \inf_{[1+\epsilon,2]} f(x) = 0 \qquad M_3 = \sup_{[1+\epsilon,2]} f(x) = 0$$

and we compute the lower and upper sum:

$$L(f, \overline{P}) = m_1 [(1 - \epsilon) - 0] + m_2 [(1 + \epsilon) - (1 + \epsilon)] + m_3 [2 - (1 + \epsilon)] = 1 - \epsilon$$
$$U(f, \overline{P}) = M_1 [(1 - \epsilon) - 0] + M_2 [(1 + \epsilon) - (1 + \epsilon)] + M_3 [2 - (1 + \epsilon)] = 1 + \epsilon.$$

In conclusion,

$$\inf_P U(f, P) - \sup_P L(f, P) < U(f, \overline{P}) - L(f, \overline{P}) = 2\epsilon$$

Since ϵ can be arbitrarily small, we have the equality $\sup_P L(f, P) = \inf_P U(f, P) = 1$ and therefore $f \in \mathcal{R}([a, b])$ and $\int_0^2 f(x)\mathrm{d}x = 1$.

2 The class of functions $\mathcal{R}([a, b])$

How can we spot an integrable function? The following theorem gives us some sufficient condition for a function to be (Riemann) integrable.

Theorem 11. *Let $f : [a, b] \to \mathbb{R}$ be a bounded function.*

a) If f is continuous on $[a, b]$, then $f \in \mathcal{R}([a, b])$.

b) If f is monotone on $[a, b]$, then $f \in \mathcal{R}([a, b])$.

Proof. $a)$ f is continuous on a closed bounded interval $[a, b]$, therefore f is uniformly continuous, meaning that $\forall\, \epsilon > 0$ there exists $\delta = \delta(\epsilon)$ such that $|f(x) - f(y)| < \epsilon$ for $|x - y| < \delta$ $(x, y \in [a, b])$.

Consider a partition P of $[a, b]$ such that the distance between the points is smaller than δ: $\Delta x_i < \delta$ for all $i = 1, \ldots, n$. On the other hand, f is continuous on each interval $[x_{i-1}, x_i]$, therefore it achieves its maximum and minimum:

$$f(t_i) = \max_{[x_{i-1}, x_i]} f(x) \qquad f(s_i) = \min_{[x_{i-1}, x_i]} f(x)$$

for some points $t_i, s_i \in [x_{i-1}, x_i]$, for all $i = 1, \ldots, n$.

Calculating the lower and upper sums, we have

$$U(f, P) - L(f, P) = \sum_{i=1}^{n} [f(t_i) - f(s_i)]\, \Delta x_i < \epsilon \sum_{i=1}^{n} \Delta x_i = \epsilon(b - a)$$

The statement follows from point $b)$ of Proposition 8.

b) Assume f is increasing (for f decreasing the argument is the same). Consider the partition $P^{(n)}$ which divides the interval $[a,b]$ into n sub-intervals of equal length (and the length is $\frac{b-a}{n}$), meaning $x_i = a + i\frac{b-a}{n}$, $i = 0, \dots, n$.

Then, since f is increasing, we have $m_i = \inf_{[x_{i-1},x_i]} f(x) = f(x_{i-1})$ and $M_i = \sup_{[x_{i-1},x_i]} f(x) = f(x_i)$; the lower and upper sums are

$$\begin{aligned} U(f,P) - L(f,P) &= \sum_{i=1}^{n} \left[f(x_i) - f(x_{i-1})\right] \Delta x_i = \sum_{i=1}^{n} \left[f(x_i) - f(x_{i-1})\right] \frac{b-a}{n} \\ &= \frac{b-a}{n} \sum_{i=1}^{n} \left[f(x_i) - f(x_{i-1})\right] = \frac{b-a}{n} \left[f(b) - f(a)\right] \longrightarrow 0 \end{aligned}$$

as long as $n \nearrow +\infty$. ☐

More generally, we have the following theorem

Theorem 12. *Let $f : [a,b] \to \mathbb{R}$ be a bounded function with finitely many discontinuities (i.e. f is continuous on $[a,b]$ except on a finite number of points), then $f \in \mathcal{R}([a,b])$.*

Sketch of the proof. Divide the interval $[a,b]$ into finitely many subintervals $[a_{i-1}, a_i]$ where f is continuous on the interior: $[a,b] = [a_1,a_2] \cup [a_2,a_3] \cup \ldots \cup [a_{n-1},a_n]$. Then, $f \in \mathcal{R}([a_{i-1},a_i])$ for all the subintervals $[a_{i-1},a_i]$ and by using the additivity result with respect to the domain of integration (see Proposition 17), we have

$$\int_a^b f(x)\mathrm{d}x = \sum_i \int_{a_{i-1}}^{a_i} f(x)\mathrm{d}x$$

therefore $f \in \mathcal{R}([a,b])$. ☐

The following proposition is claiming that the space of functions $\mathcal{R}([a,b])$ is a vector space on $\mathbb{R}$, indeed the map that associates to a function $f \in \mathcal{R}([a,b])$ the number $\int_a^b f(x)\mathrm{d}x$ is linear (additive a) and homogeneous b)).

Proposition 13. *Let $f, g \in \mathcal{R}([a,b])$ and $\alpha \in \mathbb{R}$, then*

a) $f + g \in \mathcal{R}([a,b])$ *and*

$$\int_a^b \left[f(x) + g(x)\right] \mathrm{d}x = \int_a^b f(x)\mathrm{d}x + \int_a^b g(x)\mathrm{d}x$$

b) $\alpha f \in \mathcal{R}([a,b])$ *and*

$$\int_a^b \left[\alpha f(x)\right] \mathrm{d}x = \alpha \int_a^b f(x)\mathrm{d}x$$

Proof. a)

This proof is left as an exercise. Just be careful, because in general we have that

$$\begin{aligned} &\sup_P L(f+g,P) \leq \sup L(f,P) + \sup_P L(g,P) \\ &\inf_P L(f+g,P) \geq \inf L(f,P) + \inf_P L(g,P) \end{aligned}$$

for f, g arbitrary bounded functions.

b)

Consider first $\alpha \geq 0$. Let P be a partition of $[a, b]$:

$$\inf \{\alpha f(x) \,|\, x \in [x_{i-1}, x_i]\} = \alpha \inf \{f(x) \,|\, x \in [x_{i-1}, x_i]\} = \alpha m_i$$

because $\alpha \geq 0$. Therefore,

$$L(\alpha f, P) = \sum_i \alpha m_i \Delta x_i = \alpha \sum_i m_i \Delta x_i = \alpha L(f, P).$$

Similarly, $U(\alpha f, P) = \alpha U(f, P)$.

In conclusion, $(f \in \mathcal{R}([a, b]))$

$$\sup_P L(\alpha f, P) = \sup_P \{\alpha L(f, P)\} = \alpha \sup_P L(f, P) = \alpha \inf_P U(f, P) = \inf_P \{\alpha U(f, P)\} = \inf_P U(\alpha f, P)$$

which implies that $\alpha f \in \mathcal{R}([a, b])$.

To prove the same result for $\alpha < 0$ it is sufficient to prove it for $\alpha = -1$, meaning that we need to show that if $f \in \mathcal{R}([a, b])$, then $-f \in \mathcal{R}([a, b])$.

This follows by the fact that

$$\begin{aligned} \inf \{-f(x) \,|\, x \in [x_{i-1}, x_i]\} &= -\sup \{f(x) \,|\, x \in [x_{i-1}, x_i]\} = -M_i \\ \sup \{-f(x) \,|\, x \in [x_{i-1}, x_i]\} &= -\inf \{f(x) \,|\, x \in [x_{i-1}, x_i]\} = -m_i \end{aligned}$$

therefore,

$$L(-f, P) = \sum_i -M_i \Delta x_i = -\sum_i M_i \Delta x_i = -U(f, P).$$

Similarly, $U(-f, P) = -L(f, P)$. In conclusion, $(f \in \mathcal{R}([a, b]))$

$$\sup_P L(-f, P) = \sup_P \{-U(f, P)\} = -\inf_P U(f, P) = -\sup_P L(f, P) = \inf_P \{-L(f, P)\} = \inf_P U(-f, P)$$

which implies that $-f \in \mathcal{R}([a, b])$.

$\square$

There are a few more properties on the space $\mathcal{R}([a, b])$.

Proposition 14. *Let $f, g \in \mathcal{R}([a, b])$, f a bounded unction ($m \leq f(x) \leq M$, for all $x \in [a, b]$) and $\Phi : [m, M] \to \mathbb{R}$ a continuous function ($\Phi \in C^0([m, M])$). Then,*

a) $f \circ \Phi \in \mathcal{R}([a, b])$

b) $f \cdot g \in \mathcal{R}([a, b])$

c) $|f| \in \mathcal{R}([a, b])$ *and*

$$\left| \int_a^b f(x) \mathrm{d}x \right| \leq \int_a^b |f(x)| \, \mathrm{d}x$$

3 Properties of the integral

Clearly, if the function f has constant sign, for example if $f(x) \geq 0$ everywhere on $[a, b]$, then also $L(f, P) \geq 0$ for any partition P and therefore passing to the supremum $\sup_P L(f, P) \geq 0$. If additionally $f \in \mathcal{R}([a, b])$, then its integral will be automatically a non-negative number: $\int_a^b f(x)\mathrm{d}x = \inf_P U(f, P) = \sup_P L(f, P) \geq 0$.

From this remark, the following properties follow:

Proposition 15 (Monotonicity). *Let $f, g \in \mathcal{R}([a, b])$, then*

a) if $f(x) \geq 0$ (respectively, $f(x) \leq 0$) for all $x \in [a, b]$, then $\displaystyle\int_a^b f(x)\mathrm{d}x \geq 0$ *(resp. ≤ 0)*

b) if $f(x) \geq g(x)$ for all $x \in [a, b]$, then $\displaystyle\int_a^b f(x)\mathrm{d}x \geq \int_a^b g(x)\mathrm{d}x$

It is clear that if $f(x) = 0$ on $[a, b]$, then $f \in \mathcal{R}([a, b])$ and $\int_a^b f(x)\mathrm{d}x = \int_a^b 0\,\mathrm{d}x = 0$. On the other hand, the reverse is not true in general. Consider for example the function

$$g(x) = \begin{cases} 1 & x \in [0, 2] \\ -1 & x \in [-2, 0) \end{cases}$$

then $g \in \mathcal{R}([a, b])$ and $\int_{-2}^{2} g(x)\mathrm{d}x = 0$

If we require the function to be continuous everywhere on the domain $[a, b]$ and with constant sign, then the following theorem holds.

Theorem 16 (Nullifying theorem). *Let $f : [a, b] \to \mathbb{R}$ continuous and with constant sign. If $\int_a^b f(x)\mathrm{d}x = 0$, then $f(x) = 0$ everywhere on $[a, b]$.*

Proof. Assume $f(x) \geq 0$ on $[a, b]$ (the argument is the same for $f(x) \leq 0$). Suppose there exists a point $y \in [a, b]$ such that $f(y) > 0$: by continuity, there exists a positive number $\alpha \in \mathbb{R}_+$ such that $f(x) > \frac{f(y)}{2} > 0$ for $x \in (y - \alpha, y + \alpha) \subseteq [a, b]$.

Define the function $g : [a, b] \to \mathbb{R}$ as

$$g(x) = \begin{cases} \frac{f(y)}{2} & x \in (y - \alpha, y + \alpha) \\ 0 & \text{everywhere else in } [a, b] \end{cases}$$

g is integrable on $[a, b]$ because it's bounded and it has only two points of discontinuities (at $y + \alpha$ and $y - \alpha$); moreover, $f(x) \geq g(x)$ by construction. Therefore,

$$0 = \int_a^b f(x)\mathrm{d}x \geq \int_a^b g(x)\mathrm{d}x = \frac{f(y)}{2}(y + \alpha - y + \alpha) = \frac{f(y)}{2} 2\alpha = \alpha f(y) > 0$$

and we reached a contradiction. □

Proposition 17 (Additivity with respect to the interval of integration). *Let $f : [a, b] \to \mathbb{R}$ and $c \in (a, b)$, then $f \in \mathcal{R}([a, b])$ if and only if $f \in \mathcal{R}([a, c])$ and $f \in \mathcal{R}([c, b])$ and the following holds*

$$\int_a^b f(x)\mathrm{d}x = \int_a^c f(x)\mathrm{d}x + \int_c^b f(x)\mathrm{d}x.$$

Therefore, the restriction of an integrable function $f \in \mathcal{R}([a,b])$ to a sub-interval, still gives an integrable function $f \in \mathcal{R}([c,d])$ where $[c,d] \subset [a,b]$.

Proof. Consider a partition $P_1 = \{x_0, \ldots, x_k\}$ of $[a,c]$ and a partition $P_2 = \{x_k, \ldots, x_n\}$ of $[c,b]$, then $P = P_1 \cup P_2$ is a partition of the whole interval $[a,b]$:

$$L(f,P) = \sum_{i=1}^{n} m_i \Delta x_i = \sum_{i=1}^{k} m_i \Delta x_i + \sum_{i=k+1}^{n} m_i \Delta x_i = L(f,P_1) + L(f,P_2)$$

Take the supremum on both sides. Clearly, sup over all partitions of $[a,b]$ that also contain the point c (as point of the partition) is smaller than taking the supremum over any kind of partition of $[a,b]$: $\sup_{P;c\in P} L(f,P) \le \sup_P L(f,P)$. On the other hand, if we have a partition P that doesn't contain c, then we can consider its refinement $\widetilde{P} = P \cup \{c\}$ and we have $L(f,P) \le L(f,\widetilde{P})$, therefore $\sup_P L(f,P) \le \sup_{P;c\in P} L(f,P)$ and we get the equality

$$\begin{gathered}\sup_P L(f|_{[a,b]},P) = \sup_{P;c\in P} L(f|_{[a,b]},P) = \\ = \sup\left\{L(f|_{[a,c]},P_1) + L(f|_{[c,b]},P_2) \,|\, P_1 = \text{partition of } [a,c],\ P_2 = \text{partition of } [c,b]\right\} = \\ = \sup\left\{L(f|_{[a,c]},P_1) \,|\, P_1 = \text{partition of } [a,c]\right\} + \sup\left\{L(f|_{[c,b]},P_2) \,|\, P_2 = \text{partition of } [c,b]\right\}\end{gathered}$$

The same argument can be used to prove that

$$\begin{gathered}\inf_P U(f|_{[a,b]},P) = \\ = \inf\left\{U(f|_{[a,c]},P_1) \,|\, P_1 = \text{partition of } [a,c]\right\} + \inf\left\{U(f|_{[c,b]},P_2) \,|\, P_2 = \text{partition of } [c,b]\right\}\end{gathered}$$

Now we can easily prove the double implication.

($\Rightarrow$)

If $f \in \mathcal{R}([a,b])$, then

$$\begin{gathered}\inf_{P_1} U(f|_{[a,c]},P_1) + \inf_{P_2} U(f|_{[c,b]},P_2) = \inf_P U(f|_{[a,b]},P) = \sup_P L(f|_{[a,b]},P) = \\ = \sup_{P_1} L(f|_{[a,c]},P_1) + \sup_{P_2} L(f|_{[c,b]},P_2)\end{gathered}$$

On the other hand, in general we have

$$\sup_{P_1} L(f|_{[a,c]},P_1) \le \inf_{P_1} U(f|_{[a,c]},P_1) \qquad \text{and} \qquad \sup_{P_2} L(f|_{[c,b]},P_2) \le \inf_{P_2} U(f|_{[c,b]},P_2)$$

implying the equality

$$\begin{aligned}\sup_{P_1} L(f|_{[a,c]},P_1) &= \inf_{P_1} U(f|_{[a,c]},P_1) = \int_a^c f(x)\mathrm{d}x \\ \sup_{P_2} L(f|_{[c,b]},P_2) &= \inf_{P_2} U(f|_{[c,b]},P_2) = \int_c^b f(x)\mathrm{d}x\end{aligned}$$

and the formula $\int_a^b f(x)\mathrm{d}x = \int_a^c f(x)\mathrm{d}x + \int_c^b f(x)\mathrm{d}x$ holds.

($\Leftarrow$)

If $f \in \mathcal{R}([a,c])$ and $f \in \mathcal{R}([c,b])$, then

$$\inf_P U(f|_{[a,b]}, P) = \inf_{P_1} U(f|_{[a,c]}, P_1) + \inf_{P_2} U(f|_{[c,b]}, P_2) = \sup_{P_1} L(f|_{[a,c]}, P_1) + \sup_{P_2} L(f|_{[c,b]}, P_2) =$$
$$= \sup_P L(f|_{[a,b]}, P)$$

therefore $f \in \mathcal{R}([a,b])$ and the formula $\int_a^b f(x)\mathrm{d}x = \int_a^c f(x)\mathrm{d}x + \int_c^b f(x)\mathrm{d}x$ holds. □

Theorem 18 (Mean Value Theorem for Riemann Integral). *Given $f \in \mathcal{R}([a,b])$, then $\exists\ \gamma \in \mathbb{R}$ such that*

$$\int_a^b f(x)\mathrm{d}x = \gamma(b-a)$$

where

$$m = \inf_{[a,b]} f(x) \leq \gamma \leq \sup_{[a,b]} f(x) = M$$

In particular, if $f \in C^0([a,b])$, then there exists $c \in (a,b)$ such that $\gamma = f(c)$.

Proof. Since f is bounded ($m \leq f(x) \leq M$ for all $x \in [a,b]$) and by monotonicity (Proposition 15), we have

$$m(b-a) = \int_a^b m\mathrm{d}x \leq \int_a^b f(x)\mathrm{d}x \leq \int_a^b M\mathrm{d}x = M(b-a)$$

Therefore the number

$$\gamma := \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x$$

□

is a real number such that $m \leq \gamma \leq M$. Furthermore, if f is continuous, the intermediate value theorem guarantees that there exists at least one point $c \in [a,b]$ such that $f(c) = \gamma$.

So far, we always assumed that $a < b$ for the intervals of integration. In general, for any $f \in \mathcal{R}([a,b])$, we set

- $\displaystyle\int_a^a f(x)\mathrm{d}x = 0$
- $\displaystyle\int_b^a f(x)\mathrm{d}x = -\int_a^b f(x)\mathrm{d}x$

4 Integral function and its properties

Consider a function $f : [a,b] \to \mathbb{R}$, integrable on $[a,b]$ and fix a point $x_0 \in [a,b]$.

Definition 19. The **integral function** of f with base point x_0 is the function $F_{x_0} : [a,b] \to \mathbb{R}$ defined as

$$F_{x_0}(x) = \int_{x_0}^x f(t)\mathrm{d}t$$

If we fix another point $x_1 \in [a,b]$, then we get a new integral function F_{x_1} with this new base point. What is the relationship between F_{x_0} and F_{x_1}?

$$F_{x_1}(x) = \int_{x_1}^{x} f(t)\mathrm{d}t = \int_{x_1}^{x_0} f(t)\mathrm{d}t + \int_{x_0}^{x} f(t)\mathrm{d}t = C + F_{x_0}(x)$$

where we set $C = \int_{x_1}^{x_0} f(t)\mathrm{d}t$ (f is integrable on $[a,b]$, therefore it's integrable on the smaller interval with endpoints x_1 and x_0).

This means that two integral functions of the same function f but with different base point differ by a constant. If we want to study the properties of this type of functions is sufficient to study just one of them, say $F_a(x)$ (the integral function with base point the left endpoint $x = a$).

From now on, we will denote simply by F the integral function of f with base point a.

Theorem 20 (Fundamental theorem of calculus). *Let $f \in \mathcal{R}([a,b])$, then the integral function*

$$F(x) = \int_a^x f(x)\mathrm{d}x$$

is uniformly continuous on $[a,b]$.

Moreover, if f is also continuous in a point $c \in [a,b]$, then F is differentiable in that point c and we have

$$F'(c) = f(c).$$

Proof. Let $x, y \in [a,b]$. We know that f is bounded ($\exists\, K > 0$ such that $|f(x)| \leq K$ for all $t \in [a,b]$):

$$|F(x) - F(y)| = \left|\int_a^x f(t)\mathrm{d}t - \int_a^y f(t)\mathrm{d}t\right| = \left|\int_x^y f(t)\mathrm{d}t\right| \leq \left|\int_x^y |f(t)|\,\mathrm{d}t\right| \leq \left|K\int_x^y \mathrm{d}t\right| = K|x-y| \to 0$$

if $y \to x$; therefore, F is continuous on $[a,b]$.

Clearly, for any ϵ, we can pick $\delta = \epsilon$ (and this choice does not depend on x, y) and we get that

$$|F(x) - F(y)| \leq K|x-y| < K\delta = K\epsilon,$$

implying that F is actually uniformly continuous on $[a,b]$.

Assume now that f is continuous at a point $c \in [a,b]$: this means that $\forall\, \epsilon > 0\ \exists\, \delta = \delta(\epsilon, c) > 0$ such that $|f(t) - f(c)| < \epsilon$ for $|t - c| < \delta$. We write now the increment ratio of F:

$$\frac{F(x)-F(c)}{x-c} = \frac{1}{x-c}\left[\int_a^x f(t)\mathrm{d}t - \int_a^c f(t)\mathrm{d}t\right] = \frac{1}{x-c}\int_c^x f(t)\mathrm{d}t =$$
$$= \frac{1}{x-c}\int_c^x [f(t) + f(c) - f(c)]\,\mathrm{d}t = \frac{1}{x-c}\int_c^x [f(t) - f(c)]\,\mathrm{d}t + \frac{1}{x-c}\int_c^x f(c)\mathrm{d}t =$$
$$= \frac{1}{x-c}\int_c^x [f(t) - f(c)]\,\mathrm{d}t + f(c)$$

Now given $\epsilon > 0$, for all x such that $|x - c| < \delta$ (which implies that also all t between c and x] are also such that $|t - c| < \delta$)

$$\left|\frac{F(x)-F(c)}{x-c} - f(c)\right| = \left|\frac{1}{x-c}\int_c^x [f(t)-f(c)]\,\mathrm{d}t\right| \leq \frac{1}{|x-c|}\left|\int_c^x |f(t)-f(c)|\,\mathrm{d}t\right| <$$
$$< \frac{\epsilon}{|x-c|}\left|\int_c^x \mathrm{d}t\right| = \epsilon \to 0$$

therefore F is differentiable in c and we have that $F'(c) = f(c)$. □

The theorem claims that if f is continuous everywhere on $[a, b]$, then its integral function F is a primitive (antiderivative) of f.

Definition 21. A function $F : [a, b] \to \mathbb{R}$ is called **primitive** or **antiderivative** of a function $f : [a, b] \to \mathbb{R}$ if F is differentiable on $[a, b]$ and $F'(x) = f(x)$ for all $x \in [a, b]$.

Theorem 22 (**Fundamental theorem of calculus – part II**). *Let $f : [a, b] \to \mathbb{R}$ be a continuous function and let ϕ be an antiderivative of f on $[a, b]$. Then,*

$$\int_a^b f(t)\mathrm{d}t = \phi(b) - \phi(a).$$

Proof. Since f is continuous, then f is integrable on $[a, b]$ and its integral function $F(x) = \int_a^x f(t)\mathrm{d}t$ is an antiderivative of f. Since ϕ is another antiderivative of f, ϕ differs from F by a constant:

$$\phi(x) = F(x) + C = \int_a^x f(t)\mathrm{d}t + C \quad \forall\, x \in [a, b].$$

Setting $x = a$ in the equation above we get $C = \phi(a)$ and setting $x = b$ we get the statement:

$$\phi(b) = \int_a^b f(t)\mathrm{d}t + \phi(a).$$

□

We discuss now the two main strategies for calculating an integral: integration by parts and change of variable.

Proposition 23 (**Integration by parts**). *Let $f, g : [a, b] \to \mathbb{R}$ such that $f, g \in C^1([a, b])$. Then,*

$$\int_a^b f(t)g'(t)\mathrm{d}t = f(b)g(b) - f(a)g(a) - \int_a^b f'(t)g(t)\mathrm{d}t$$

Sketch of the proof. Apply Leibniz rule for the derivative of the product. □

Regarding the “change of variable” technique, we need to first establish some reasonable hypothesis: if we set $s = g(t)$, where the variable $t \in [a, b]$ and the new (dependent) variable s varies in a new interval, say $[\alpha, \beta]$, we need the mapping $t \mapsto g(t) = s$ to be a bijection between $[a, b]$ and $[\alpha, \beta]$; moreover, we need to require that if such map is regular on $[a, b]$ (say, it's a continuous function with continuous derivative), also its inverse has the same regularity.

Such properties are certainly satisfied if we assume the following hypothesis:

Proposition 24 (**Change of variable**). *Let $f : [\alpha, \beta] \to \mathbb{R}$ be a continuous function and let $g : [a, b] \to \mathbb{R}$ be a $C^1([a, b])$-function with derivative $g'(t) \neq 0$ for all $t \in [a, b]$ and $g([a, b]) = [\alpha, \beta]$. Then*

$$\int_a^b f(g(t))g'(t)\mathrm{d}t = \int_{g(a)}^{g(b)} f(s)\mathrm{d}s.$$

Proof. First of all we notice that since f, g, g' are continuous, then $(g \circ f) \cdot g' \in \mathcal{R}([a, b])$. Consider the integral function

$$F(y) = \int_{g(a)}^{y} f(s)\mathrm{d}s$$

by the Fundamental theorem of calculus we know that F is differentiable and $F'(y) = f(y)$ for all $y \in [g(a), g(b)]$. We apply now the chain rule: if $y = g(t)$, then

$$[F(g(t))]' = F'(g(t))g'(t) = f(g(t))g'(t);$$

also $F(g(a)) = \int_{g(a)}^{g(a)} f(s)\mathrm{d}s = 0$, therefore

$$\int_{g(a)}^{g(b)} f(s)\mathrm{d}s = F(g(b)) = F(g(b)) - F(g(a)) = \int_{a}^{b} [F(g(t))]' \,\mathrm{d}t = \int_{a}^{b} f(g(t))g'(t)\mathrm{d}t.$$

$\square$