Notes on Series

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MATH 317-01 Advanced Calculus of one variable

1 Definition

Definition 1 (formal). Given a sequence $\{x_n\}_{n=1}^{+\infty}$, we define a **series** as the formal sum of all its elements and we use the following notation:

$$\sum_{n=1}^{+\infty} x_n.$$

In general, a series is just a formal object, a symbol. We can then wonder whether this symbol represents a real quantity (a number), namely if we can indeed sum all the elements of a given sequence $\{x_n\}$.

Definition 2. Given a sequence $\{x_n\}$, we can define the **partial sum** of order n as the sum of all the first n elements of the sequence:

$$s_n := \sum_{k=1}^{n} x_k.$$

i.e.

$$\begin{aligned} s_1 &= x_1 \\ s_2 &= x_1 + x_2 \\ s_3 &= x_1 + x_2 + x_3 \\ s_4 &= x_1 + x_2 + x_3 + x_4 \\ &\vdots \end{aligned}$$

Remark 3. *We can observe that there is a one-to-one correspondence between the elements of the sequence $\{x_n\}$ and the partial sums, since any partial sum of order n is constructed iteratively from the previous partial sum of order $n-1$ and the term x_n:*

$$s_n = x_1 + x_2 + \ldots + x_{n-1} + x_n = s_{n-1} + x_n \qquad \Leftrightarrow \qquad x_n = s_n - s_{n-1} \quad \forall\, n \geq 2$$

This means that if we only know the sequence of partial sums $\{s_n\}$ we can still recover the original sequence $\{x_n\}$ by calculating each term as $x_1 = s_1$ and $x_n = s_n - s_{n-1}$ for $n \geq 2$.

Definition 4 (rigorous). Given a sequence $\{x_n\}$, a **series with general term** x_n is denoted as $\sum_{n=1}^{+\infty} x_n$ and it is defined as the limit of the sequence of its partial sums:

$$\sum_{n=1}^{+\infty} x_n := \lim_{n\to+\infty} s_n.$$

A series is **convergent** if $\lim_{n\to+\infty} s_n = S$ exists and it's a real number S which will be called **sum of the series**. If $\lim_{n\to+\infty} s_n = \pm\infty$, then the series is said to be **divergent**. These two types of series are generically called **regular** series.

If $\lim_{n\to+\infty} s_n$ does not exist, the series is called **irregular**.

Note 5. *Sometimes it might be more convenient to shift the starting point of the index from* 1 *to* 0 *or another number. For example*

$$\sum_{n=1}^{+\infty} \frac{1}{2^{n-1}} = \sum_{k=0}^{+\infty} \frac{1}{2^k} = 1 + \frac{1}{2} + \frac{1}{4} + \dots$$

Example 1. Consider the constant sequence $\{x_n = c\}$ ($c \in \mathbb{R}$), then the sequence of partial sums has general term

$$s_n = c + c + c + \dots + c = nc;$$

therefore,

$$\lim_{n\to+\infty} s_n = \lim_{n\to+\infty} nc = \begin{cases} +\infty & \text{if } c > 0 \\ -\infty & \text{if } c < 0 \\ 0 & \text{if } c = 0 \end{cases}$$

The series $\sum_{n=1}^{\infty} c$ is convergent if $c = 0$ and divergent in all the other cases.

Example 2. Consider the alternating sequence $\{x_n = (-1)^n\}$, the the partial sum of order n is equal to

$$s_n = -1 + 1 - 1 + 1 - 1 \dots = \begin{cases} -1 & \text{if } n = \text{odd} \\ 0 & \text{if } n = \text{even} \end{cases}$$

therefore $\lim_{n\to+\infty} s_n$ does not exists and the series $\sum_{n=1}^{\infty} (-1)^n$ is irregular.

Exemple 3. Consider the sequence $\{x_n = n\}$. The partial sums are given as

$$s_n = 1 + 2 + 3 + \dots + (n-1) + n = \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

and clearly

$$\lim_{n\to+\infty} s_n = \lim_{n\to+\infty} \frac{n(n+1)}{2} = +\infty$$

therefore the series $\sum_{n=1}^{+\infty} n$ is divergent.

In these three examples it was easy to explicitly calculate the sequence of partial sums and its limit, but this is not always the case. We will see later on more general "convergence test" to establish the behaviour of a series.

2 General properties

The main aspects that one wishes to study about a series are

1) to determine what is the behaviour of the series, i.e. whether it is convergent, divergent or irregular;

2) in the case where the series it's convergent, to calculate the sum.

For most of the cases, already establishing the first bullet is a hard task. Determine the sum of a series is usually even harder, if not impossible with the tools we know right now.

Proposition 6. *If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are two series with the same general term, except for a finite number of indexes, then the two series have the same behaviour.*

Proof. If the series differ by a finite number of indexes, there exist an index N such that for all the indexes after N the terms are the same: $a_n = b_n \ \forall \ n \geq N$.

Therefore, if we consider the (respective) partial sums $s_n = \sum_{k=1}^{n} a_k$ and $t_n := \sum_{k=1}^{n} t_k$ and calculate their difference

$$s_{N+m} - t_{N+m} = \sum_{k=1}^{N+m} a_k - \sum_{k=1}^{N+m} b_k = \sum_{k=1}^{N+m} a_k - b_k = \sum_{k=1}^{N} a_k - b_k =: \kappa \ \forall \ m \geq 0$$

This implies that the sequence $\{s_n - t_n\}_{n=1}^{\infty}$ is constant (equal to κ), up to a finite number of terms; its limit exists $\lim_{n\to+\infty} s_n - t_n = \kappa$ and the respective series have the same behaviour. $\square$

Note 7. *Even if both of the series in the proposition above happen to be convergent, their sum is different!*

We will see now a first **necessary** condition for a series to be convergent.

Proposition 8. *If the series $\sum_{n=1}^{\infty} x_n$ is convergent, then $\lim_{n\to+\infty} x_n = 0$.*

Proof. Given that the series is convergent, we denote by S its sum: $S := \sum_{n=1}^{\infty} x_n = \lim_{n\to+\infty} s_n$, meaning that the sequence of partial sums converges to S. Then, we have

$$x_n = s_n - s_{n-1} \to S - S = 0.$$

$\square$

Examples.

1) We saw already that the series $\sum_{n=1}^{\infty}(-1)^n$ is irregular. Indeed, its general term is $(-1)^n$ which does not converge to 0.

2) Also the series $\sum_{n=1}^{\infty} 2^{-\frac{1}{n}}$ does not converge because $2^{-\frac{1}{n}} = \frac{1}{2^{\frac{1}{n}}} \to 1 \neq 0$.

Be careful, though! The proposition above only gives a criterion to quickly spot some series that are **not** convergent (in particular, the ones whose general term doesn't go to zero); on the other hand, if we are given a series with general term that goes to zero, we still cannot conclude that the series is convergent.

Counterexamples.

1) consider the Harmonic Series
$$\sum_{n=1}^{\infty} \frac{1}{n}.$$
Although its general term is $\frac{1}{n} \to 0$ as $n \to +\infty$, the series is divergent.
To see this, suppose it's convergent and call S the sum of the series. Then,
$$\begin{aligned}
S &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \dots \\
&> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{6} + \frac{1}{6}\right) + \left(\frac{1}{8} + \frac{1}{8}\right) + \dots = 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots \\
&= \frac{1}{2} + S
\end{aligned}$$
this leads to $S > \frac{1}{2} + S$, which is a contradiction.

2) Consider the series
$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1} + \sqrt{n}}.$$
Also in this case the general term $x_n = \frac{1}{\sqrt{n+1}+\sqrt{n}} \to 0$, but the series is not convergent. Indeed,
$$\frac{1}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}} \frac{\sqrt{n+1} - \sqrt{n}}{\sqrt{n+1} - \sqrt{n}} = \frac{\sqrt{n+1} - \sqrt{n}}{n+1-n} = \sqrt{n+1} - \sqrt{n}$$
and the partial sums are telescopic sums (meaning, all the terms in the summation cancel out except the first one and the last one)
$$\begin{aligned}
s_n &= \left(\sqrt{2} - \sqrt{1}\right) + \left(\sqrt{3} - \sqrt{2}\right) + \left(\sqrt{4} - \sqrt{3}\right) + \dots + \left(\sqrt{n} - \sqrt{n-1}\right) + \left(\sqrt{n+1} - \sqrt{n}\right) \\
&= \sqrt{n+1} - 1 \to +\infty \quad \text{as } n \to +\infty.
\end{aligned}$$

A **necessary and sufficient** condition for a series to be convergent is the Cauchy condition.

Definition 9. A series $\sum_{n=1}^{\infty} x_n$ is **Cauchy** if the sequence of partial sums $\{s_n\}$ is a Cauchy sequence.

Proposition 10 (Cauchy criterion). *A series $\sum_{n=1}^{\infty} x_n$ is convergent $\Leftrightarrow$ the series is Cauchy.*

Proof. By definition of a convergent series, $\sum_{n=1}^{\infty} x_n$ is convergent $\Leftrightarrow$ the sequence of partial sums $\{s_n\}$ converges. We also know that a convergent sequence is equivalent to say that the sequence is a Cauchy sequence, therefore (by the definition above) the series is Cauchy. $\square$

Corollary 11. A (Cauchy) series is such that $\forall\ \epsilon > 0\ \exists M_\epsilon \in \mathbb{N}$ such that $\forall\ n, m \geq M_\epsilon$ we have
$$|s_n - s_m| < \epsilon,$$
meaning (assume without lost of generality that $n > m$)
$$|s_n - s_m| = \left| \sum_{k=1}^{n} x_k - \sum_{k=1}^{m} x_n \right| = \left| \sum_{k=m+1}^{n} x_n \right| < \epsilon.$$

The Cauchy criterion is not very handy to use in practical situations. Therefore, we will now see other conditions (only **sufficient**) that can guarantee convergence of a series.

3 Absolute convergence

Definition 12. A series $\sum_{n=1}^{\infty} x_n$ is **absolute convergent** if $\sum_{n=1}^{\infty} |x_n|$ is convergent.

Remark 13. *If a series $\sum_{n=1}^{\infty} x_n$ converges, but not absolutely (i.e. the series of the absolute values of its terms doesn't converge), then we say that the series converges* **simply**.

Remark 14. *If the general terms of the series are non-negative ($x_n \geq 0 \ \forall\ n \geq 1$), then $\sum_{n=1}^{\infty} x_n = \sum_{n=1}^{\infty} |x_n|$ and the concept of simple convergence ad absolute converges coincide.*

Proposition 15. *Absolute convergence implies simple convergence, meaning that if $\sum_{n=1}^{\infty} |x_n|$ converges, then also $\sum_{n=1}^{\infty} x_n$ converges.*

Proof. If $\sum_{n=1}^{\infty} |x_n|$, this means that the series satisfies the Cauchy condition: $\forall\ \epsilon > 0\ \exists M_\epsilon \in \mathbb{N}$ such that $\forall\ n, m \geq M_\epsilon$

$$\left| \sum_{m+1}^{n} |x_n| \right| < \epsilon.$$

On the other hand, by triangle inequality,

$$\left| \sum_{m+1}^{n} x_n \right| \leq \left| \sum_{m+1}^{n} |x_n| \right| < \epsilon,$$

meaning that also the series $\sum_{n=1}^{\infty} x_n$ is Cauchy, therefore convergent. □

Examples.

1) The series
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$
is simply convergent (see Leibniz criterion later in the notes), but it's not absolutely convergent because the series of absolute values is the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$, which we know to be divergent.

2) The series
$$\sum_{n=1}^{\infty} \frac{\cos(n)}{n^2}$$
where the general term has a sign that is difficult to identify (it varies according to the cosine) is absolutely convergent (see Integral test later in the notes).

4 Constant sign series

Let's assume that the general term of a series is always non-negative: $x_n \geq 0$ for all $n \in \mathbb{N}$ (all the arguments below can be analogously carried out in the case $x_n \leq 0$).

In this case we can already notice that the sequence of partial sums $\{s_n\}$ is monotone increasing and therefore $\{s_n\}$ is convergent $\Leftrightarrow$ $\{s_n\}$ is bounded.

Also, in this case, the series $\sum_{n=1}^{\infty} x_n$ can only be regular, i.e. either convergent or divergent to $\pm\infty$.

We will see now a list of convergence test that will allow us to establish the behaviour of a series. It is follows that if a series $\sum_{n=1}^{\infty} x_n$ has general term x_n with non-constant sign, we can always inspect whether the series is absolutely convergent by considering the series of absolute values $\sum_{n=1}^{\infty} |x_n|$ and this series is clearly of constant (positive) sign.

Proposition 16 (Comparison test). *Consider two sequences $\{a_n\}$ and $\{b_n\}$ such that $0 \leq a_n \leq b_n$ for all $n \in \mathbb{N}$, then*

1) if $\sum_{n=1}^{\infty} b_n$ converges, then also $\sum_{n=1}^{\infty} a_n$ converges;

2) if $\sum_{n=1}^{\infty} a_n$ diverges, then also $\sum_{n=1}^{\infty} b_n$ diverges.

Proof. Since we have the inequality $0 \leq a_n \leq b_n$, then also the partial sums have the same behaviour:

$$0 \leq s_n \leq t_n$$

and, passing to the limit, we have

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to+\infty} s_n \leq \lim_{n\to+\infty} t_n = \sum_{n=1}^{\infty} b_n.$$

We then apply the comparison test for limits of sequences. □

Corollary 17. If there exist two constant $c, C \in \mathbb{R}_+$ such that $ca_n \leq b_n \leq Ca_n$ for all $n \in \mathbb{N}$, then the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ have the same behaviour.

Proposition 18 (Integral test). *Suppose we have a function $f : [1, +\infty) \to \mathbb{R}$ such that $f(x) \geq 0$ for all $x \in [1, +\infty)$ and f is monotone decreasing over $[1, +\infty)$. Then,*

$$\int_1^{+\infty} f(x)\,\mathrm{d}x \qquad \textit{and} \qquad \sum_{n=1}^{+\infty} f(n)$$

have the same behaviour (they are both convergent of both divergent).

Proof. Consider the integral of f over the interval $[k, k+1]$ for any $k \in \mathbb{N}$:

$$\int_k^{k+1} f(x)\mathrm{d}x \leq \max_{x\in[k,k+1]} f(x) \cdot \int_k^{k+1} \mathrm{d}x = f(k) \cdot 1 = f(k)$$

and

$$\int_k^{k+1} f(x)\mathrm{d}x \geq \min_{x\in[k,k+1]} f(x) \cdot \int_k^{k+1} \mathrm{d}x = f(k+1) \cdot 1 = f(k+1),$$

therefore

$$f(k+1) \leq \int_k^{k+1} f(x)\mathrm{d}x \leq f(k)$$

and summing over the k's from 1 to $n-1$:

$$s_n - f(1) = \sum_{k=1}^{n-1} f(k+1) \leq \sum_{k=1}^{n-1} \int_k^{k+1} f(x)\mathrm{d}x = \int_1^n f(x)\mathrm{d}x \leq \sum_{k=1}^{n-1} f(k) = s_{n-1} \leq s_n$$

The last inequality follows from the fact that $f(n) \geq 0$ for all $n \in \mathbb{N}$, therefore $s_n = f(n) + s_{n-1} \geq s_{n-1}$.

Passing to the limit

$$\lim_{n\to+\infty} s_n - f(1) \leq \lim_{n\to+\infty} \int_1^n f(x)\mathrm{d}x = \int_1^{+\infty} f(x)\mathrm{d}x \leq \lim_{n\to+\infty} s_n$$

and applying the comparison test for sequences yields the result. □

5 Important examples

Geometric series: let $q \in \mathbb{R}$, define

$$\sum_{n=0}^{\infty} q^n = 1 + q + q^2 + q^3 + \ldots$$

Proposition 19. *a) If $|q| < 1$, then the series is convergent and its sum is equal to*

$$\sum_{n=0}^{\infty} q^n = \frac{1}{1-q}.$$

b) If $q \geq 1$, then the series diverges.

c) If $q \leq -1$, then the series is irregular.

Proof. If $q = 1$, then the sequence of partial sums have general term equal to $s_n = 1 + 1 + 1 + 1 + \ldots + 1 = n \cdot 1 = n$, which is clearly unbounded and diverging to $+\infty$. If $q = -1$, then the general partial sum is $s_n = -1 + 1 - 1 + 1 + \ldots$ and it is equal to zero if n is even or 1 is n is odd, therefore the sequence $\{s_n\}$ admits no limit.

For $q \neq \pm 1$, we can write the partial sums as

$$\begin{aligned} s_n = 1 + q + q^2 + q^3 + \ldots + q^n &= \frac{\left(1 + q + q^2 + q^3 + \ldots + q^n\right)(1-q)}{1-q} \\ &= \frac{1 - q^{n+1}}{1-q} \end{aligned}$$

and passing to the limit

$$\lim_{n\to+\infty} s_n = \lim_{n\to+\infty} \frac{1-q^{n+1}}{1-q} = \begin{cases} \frac{1}{1-q} & \text{if } |q| < 1 \\ +\infty & \text{if } q \geq 1 \\ \text{no limit} & \text{if } q \leq -1 \end{cases}$$

□

Generalized Harmonic Series: let $p \in \mathbb{R}$, define

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \dots$$

Proposition 20. *a) If $p > 1$, then the series is convergent.*

b) If $p \leq 1$, then the series diverges.

Proof. If $p < 0$, then the general term $\frac{1}{n^p}$ does not converges to zero, therefore the series cannot be convergent. If $p = 1$ we already know that the Harmonic series $\sum \frac{1}{n}$ is divergent. For $p > 0$ ($p \neq 1$), use the Integral test with the function $f(x) = \frac{1}{x^p}$ (f is positive and monotone decreasing on the interval $[1, +\infty)$):

$$\int_1^{+\infty} \frac{1}{x^p} \mathrm{d}x = \lim_{n \to +\infty} \int_1^n \frac{1}{x^p} \mathrm{d}x = \lim_{n \to +\infty} \left[\frac{x^{-p+1}}{-p+1} \right]_1^n = \lim_{n \to +\infty} \frac{1}{1-p} \left[n^{1-p} - 1 \right] = \begin{cases} +\infty & \text{if } 0 < p < 1 \\ \frac{1}{p-1} & \text{if } p > 1 \end{cases}$$

Therefore, the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p > 1$ and diverges for all the other cases. $\square$

Note 21. *Be careful! The value of the (indefinite) integral $\int_1^{+\infty} \frac{1}{x^p} \mathrm{d}x$ is $\frac{1}{p-1}$, but this is* **not** *the sum of the series! When using the integral test, we only care to check whether the integral converges to a number (any number) or not. The integral test doesn't give us the sum of a series.*

Remark 22. *It is possible to prove (but using much more advanced techniques that won't be shown in the present course) that*

$$\sum_{n=1}^{+\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Mengoli series:

$$\sum_{n=1}^{+\infty} \frac{1}{n(n+1)}$$

The general term can be rewritten in a more convenient form: $x_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$. This allows us to explicitly calculate the partial sums (which are again telescopic sum) and even calculate the sum of the series:

$$s_n = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \dots + \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1} \to 1$$

therefore, the Mengoli series is convergent and

$$\sum_{n=1}^{+\infty} \frac{1}{n(n+1)} = \lim_{n \to +\infty} s_n = 1.$$

<u>Another example with the Integral Test:</u> the series

$$\sum_{n=1}^{+\infty} \frac{1}{n^p (\ln n)^q}$$

is convergent for either $p > 1$ and any $q \in \mathbb{R}_+$ or for $p = 1$ and $q > 1$. To see this result, we can apply again the Integral Test with the function $f(x) = \frac{1}{x^p (\ln x)^q}$.

6 Constant sign series - part II

Proposition 23 (**Ratio Test**). *Let $\{x_n\}$ be a positive sequence ($x_n \geq 0$ for all $n \in \mathbb{N}$), then*

a) if $\limsup \frac{x_{n+1}}{x_n} < 1$, then the series $\sum_{n=1}^{\infty} x_n$ converges;

b) if $\frac{x_{n+1}}{x_n} \geq 1$ (up to finitely many indexes), then the series diverges.

Proof. a) Remember the definition of lim sup:

$$\limsup_{n\to+\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to+\infty} a_n = \lim_{n\to+\infty} \sup\left\{ \frac{x_{\ell+1}}{x_\ell} \;\middle|\; \ell \geq n \right\} < 1$$

Therefore there exists $\epsilon > 0$ such that $\limsup_{n\to+\infty} \frac{x_{n+1}}{x_n} < \epsilon < 1$. For such ϵ, there exists $M_\epsilon \in \mathbb{N}$ such that for all $n \geq M_\epsilon$

$$\frac{x_{n+1}}{x_n} \leq a_{M_\epsilon} := \sup\left\{ \frac{x_{\ell+1}}{x_\ell} \;\middle|\; \ell \geq M_\epsilon \right\} < \epsilon < 1$$

meaning that $\forall\, n \geq M_\epsilon$

$$x_n \leq \epsilon x_{n-1} < \epsilon^2 x_{n-2} < \ldots < \epsilon^{n-M_\epsilon} x_{M_\epsilon}.$$

Summing over all the terms in the sequence $\{x_n\}$, the series is bounded from above by a geometric series with general term $\epsilon < 1$ (therefore it is convergent)

$$\sum_{n=1}^{\infty} x_n = \sum_{n=1}^{M_\epsilon - 1} x_n + \sum_{n=M_\epsilon}^{\infty} x_n < \sum_{n=1}^{M_\epsilon - 1} x_n + x_{M_\epsilon} \sum_{n=M_\epsilon}^{\infty} \epsilon^{n-M_\epsilon} = K + x_{M_\epsilon} \sum_{j=0}^{\infty} \epsilon^j = K + \frac{1}{1-\epsilon} < +\infty.$$

Applying comparison test for series gives the result.

b) If $\frac{x_{n+1}}{x_n} > 1$, then $x_n < x_{n+1}$, therefore the limit of the sequence cannot be zero and the series diverges.

□

Corollary 24. Let

$$L := \limsup \frac{x_{n+1}}{x_n} \geq 0,$$

a) if $L < 1$, the series converges;

b) if $L > 1$, the series diverges.

Note 25. *Again as in the comparison test for sequences, if $L = 1$, then the test is inconclusive. For example, the generalized harmonic series $\sum \frac{1}{n^p}$ is such that $L = 1$ for any value of p (while we know that only for $p > 1$ the series converges).*

__**Exponential series:**__ let $x \in \mathbb{R}$, define

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

We want to analyze the behaviour of the series depending on $x \in \mathbb{R}$:

- for $x = 0$, the series is trivial: $\sum_{n=0}^{\infty} \frac{0^n}{n!} \equiv 0$
- for $x \neq 0$, we have two cases: either $x > 0$ and the series has constant sign, or $x < 0$ and the series does not have constant sign. We analyze both cases together by just inspecting the absolute convergence.

  We consider the series of absolute values $\sum_{n=1}^{\infty} \frac{|x|^n}{n!}$ and apply the Ratio Test:

  $$\limsup \frac{\frac{|x|^{n+1}}{(n+1)!}}{\frac{|x|^n}{n!}} = \limsup \frac{|x|}{n+1} = 0 < 1 \quad \forall\, x \in \mathbb{R} \backslash \{0\}.$$

  The series is absolutely convergent (and therefore also simply convergent) for any value of $x \in \mathbb{R}$.

Remark 26. *It is possible to prove (and we will see it later in the course) that this series is called "exponential series" because it defines the exponential function*

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad x \in \mathbb{R},$$

where e is the Nepero number.

Proposition 27 (Root Test). *Let $\{x_n\}$ be a positive sequence ($x_n \geq 0$ for all $n \in \mathbb{N}$) and let*

$$\lambda := \limsup_{n \to +\infty} \sqrt[n]{x_n} \geq 0,$$

then

a) if $\lambda < 1$, the series $\sum_{n=1}^{\infty} x_n$ converges;

b) if $\lambda > 1$, the series diverges.

Proof. a) if $\lambda < 1$, then there exists $\epsilon > 0$ such that $\lambda < \epsilon < 1$ and for such ϵ there exists $M_\epsilon \in \mathbb{N}$ such that $\forall\, n \geq M_\epsilon$ $\sqrt[n]{x_n} < \epsilon$, i.e. $x_n < \epsilon^n$.

Summing over all the elements of the sequence, the series $\sum_{n=1}^{\infty} x_n$ becomes bounded from above by a constant plus a (convergent) geometric series, therefore it is convergent as well:

$$\sum_{n=1}^{\infty} x_n = \sum_{n=1}^{M_\epsilon - 1} x_n + \sum_{n=M_\epsilon}^{\infty} x_n < K_1 + \sum_{n=M_\epsilon}^{\infty} \epsilon^n = K_1 + \sum_{n=0}^{\infty} \epsilon^n - \sum_{n=0}^{M_\epsilon - 1} \epsilon^n = K_1 + K_2 + \frac{1}{1-\epsilon} < +\infty.$$

b) if $\lambda > 1$, then there exists a subsequence $\{\sqrt[n_k]{x_{n_k}}\}$ such that $\sqrt[n_k]{x_{n_k}} \to \lambda > 1$, therefore x_{n_k} does not converges to zero. The series diverges.

$\square$

Corollary 28. Let

$$\tilde{\lambda} := \lim_{n\to+\infty} \sqrt[n]{x_n} \geq 0,$$

a) if $\tilde{\lambda} < 1$, the series converges;

b) if $\tilde{\lambda} > 1$, the series diverges.

Note 29. *Also in this case if $\tilde{\lambda} = 1$, then the test is inconclusive.*

Examples. Apply the root test to the following series to check their behaviour:

1) $\sum_{n=1}^{\infty} \left(\frac{n}{2n+1}\right)^n$:

$$\lim_{n\to+\infty} \sqrt[n]{\left(\frac{n}{2n+1}\right)^n} = \lim_{n\to+\infty} \frac{n}{2n+1} = \frac{1}{2} < 1$$

therefore the series converges.

2) $\sum_{n=1}^{\infty} \frac{(n+1)^n}{e^{3n}}$:

$$\lim_{n\to+\infty} \sqrt[n]{\frac{(n+1)^n}{e^{3n}}} = \lim_{n\to+\infty} \frac{n+1}{e^3} = +\infty$$

therefore the series diverges.

7 Alternating sign series

Definition 30. An **alternating sign series** is a series of the form

$$\sum_{n=1}^{\infty} (-1)^{n-1} x_n = x_1 - x_2 + x_3 - x_4 + \dots$$

where the term x_n is positive ($x_n > 0$ for all $n \in \mathbb{N}$).

Proposition 31 (Leibniz criterion). *Suppose that the sequence $\{x_n\}$ is such that it is*

1. *positive:* $x_n > 0$
2. *decreasing:* $x_n \geq x_{n+1}$
3. *infinitesimal:* $\lim_{n\to+\infty} x_n = 0$

Then, the alternating sign series $\sum_{n=1}^{\infty} (-1)^{n-1} x_n$ converges and the following estimate for the sum S holds: for all $n \in \mathbb{N}$

$$|s_n - S| \leq x_{n+1}.$$

Proof. Consider the odd partial sums:

$$s_{2n+1} = s_{2n-1} + (-1)^{2n-1} x_{2n} + (-1)^{2n} x_{2n+1} = s_{2n-1} - (x_{2n} - x_{2n+1}) \leq s_{2n-1}$$

since $\{x_n\}$ is decreasing. Therefore, the subsequence of odd partial sums is monotone decreasing.

On the other hand, consider the even partial sums:

$$s_{2n} = s_{2n-2} + (-1)^{2n-2} x_{2n-1} + (-1)^{2n-1} x_{2n} = s_{2n-2} + (x_{2n-1} - x_{2n}) \geq s_{2n-2}$$

Therefore, the subsequence of even partial sums is monotone increasing.

Furthermore,

$$s_{2n+1} = s_{2n} + (-1)^{2n} x_{2n+1} = s_{2n} + x_{2n+1} \geq s_{2n},$$

meaning that the sequence of partial sums is ordered in this way

$$s_2 \leq s_4 \leq \ldots \leq s_{2n} \leq s_{2n+1} \leq s_{2n-1} \leq \ldots \leq s_3 \leq s_1.$$

Therefore, both subsequences are monotone and bounded, so they are convergent. To prove that they converge to the same limit, we notice that

$$s_{2n+1} - s_{2n} = (-1)^{2n} x_{2n+1} = x_{2n+1} \to 0$$

because the sequence $\{x_n\}$ is infinitesimal. Therefore, they have the same limit: $s_{2n+1} \to S$ and $s_{2n} \to S$.

To prove the final estimate, we calculate

$$\begin{aligned} 0 &\leq S - s_n \leq s_{2n+1} - s_{2n} = x_{2n+1} \\ 0 &\leq s_{2n+1} - S \leq s_{2n+1} - s_{2n+2} = x_{2n+2}. \end{aligned}$$

□

Example. The generalized harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$ converges absolutely for $p > 1$ and it converges simply (but not absolutely) for $0 < p \leq 1$, thanks to Leibniz criterion.

8 Sum and product of series; rearrangements

Proposition 32 (Linearity). *Let $\alpha \in \mathbb{R}$ and $\sum x_n$, $\sum y_n$ convergent series.*

1. *The series $\sum_{n=1}^{\infty} \alpha x_n$ converges and $\sum_{n=1}^{\infty} \alpha x_n = \alpha \sum_{n=1}^{\infty} x_n$.*
2. *The series $\sum_{n=1}^{\infty} (x_n + y_n)$ converges and $\sum_{n=1}^{\infty} (x_n + y_n) = \sum_{n=1}^{\infty} x_n + \sum_{n=1}^{\infty} y_n$.*

Definition 33. Given two series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$, their **Cauchy product** is a new series

$$\sum_{n=0}^{\infty} c_n \quad \text{with general term } c_n = a_0 b_n + a_1 b_{n-1} + \ldots + a_n b_0 = \sum_{j=0}^{n} a_j b_{n-j}$$

Proposition 34 (**Merten's Theorem**). *If $\sum_{n=0}^{\infty} a_n$ converges absolutely and its sum is $\sum_{n=0}^{\infty} a_n = A$ and if $\sum_{n=0}^{\infty} b_n$ converges (even simply) and $\sum_{n=0}^{\infty} b_n = B$, then their Cauchy product converges as well and*

$$\sum_{n=0}^{\infty} c_n = AB.$$

Definition 35. Given a series $\sum a_n$, a **rearrangement** is a new series $\sum b_n$ obtained from the previous one by changing the summation order.

More formally, let $\sigma : \mathbb{N} \to \mathbb{N}$ be a bijection, then given $\sum_{n=1}^{\infty} a_n$ a rearrangement is the new series

$$\sum_{n=1}^{\infty} a_{\sigma(n)}.$$

The main questions that arise for these new series are: is the new series still convergent, if the original one is? What is its sum?

We report here some fundamental result without giving proofs.

Definition 36. A series $\sum a_n$ is **unconditionally convergent** if for anny rearrangement σ the new series $\sum a_{\sigma(n)}$ is still convergent and its sum is the same as the original series $\sum a_n = S = \sum a_{\sigma(n)}$.

Theorem 37 (**Dirichlet**). *A series $\sum a_n$ is unconditionally convergent if and only if it is absolutely convergent.*

Theorem 38 (**Riemann**). *If a series $\sum a_n$ is simply convergent (but not absolutely), for any $\alpha, \beta \in \mathbb{R} \cup \{\pm\infty\}$ there exists a rearrangement $\sigma : \mathbb{N} \to \mathbb{N}$ such that the new series $\sum a_{\sigma(n)}$ has the following behaviour*

$$\alpha = \liminf \sum a_{\sigma(n)} \qquad \textit{and} \qquad \beta = \limsup \sum a_{\sigma(n)}$$

Corollary 39. If a series $\sum a_n$ is simply convergent (but not absolutely), for any $\alpha \in \mathbb{R} \cup \{\pm\infty\}$ there exists a rearrangement $\sigma : \mathbb{N} \to \mathbb{N}$ such that the new series $\sum a_{\sigma(n)}$ is convergent and has sum equal to α:

$$\sum a_{\sigma(n)} = \alpha.$$

To see an example of this corollary, we refer to Example 2.6.4 from the textbook.