The Van der Pol oscillator

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MATH 3406 Differential Equations II

Contents

1 Introduction

In 1927 Balthazar Van der Pol, a Dutch electrical engineer, introduced an equation in order to describe oscillations in a vacuum tube electrical circuit. The (autonomous) equation reads

$$y'' + \mu(y^2 - 1)y' + y = 0 \qquad \mu > 0 \tag{1}$$

where $y(t)$ describes the current in a certain type of vacuum tube called "triode". The corresponding non-autonomous equation is

$$y'' + \mu(y^2 - 1)y' + y = a\sin(\omega t) \qquad \mu > 0, a, \omega \in \mathbb{R}.$$

In these notes we will mostly focused on the autonomous equation (2).

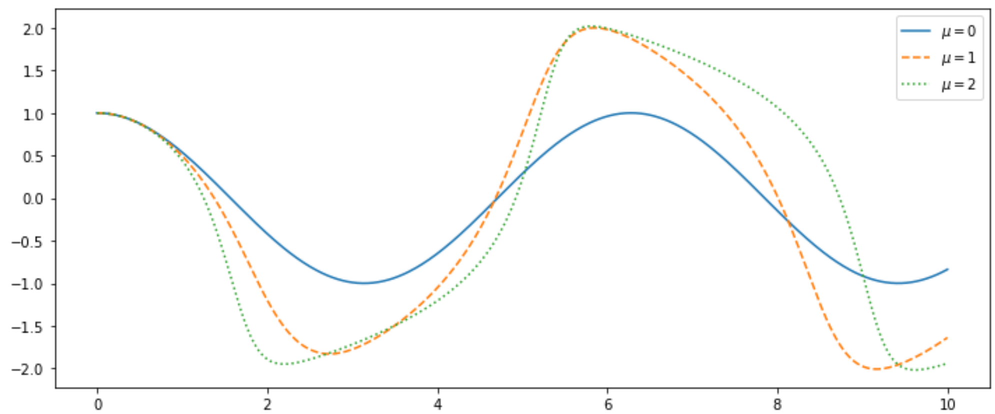

Figure 1: Solutions of the VdP equation (2) for $\mu = 0, 1, 2$.

1.1 Modelling and history - Where does the equation come from?

At the beginning of the twentieth century, vacuum tubes were used to control the flow of electricity in the circuitry of transmitters and receivers. Contemporary with Lorenz, Thompson, and Appleton, Van der Pol experimented with oscillations in a vacuum tube triode circuit and concluded that all initial conditions converged to the same periodic orbit of finite amplitude. Since this behavior is different from the behavior of solutions of linear equations, van der Pol proposed a nonlinear differential equation

$$y'' + \mu(y^2 - 1)y' + y = 0 \qquad \mu > 0 \tag{2}$$

commonly referred to as the (unforced) van der Pol equation, as a model for the behavior observed in the experiment.

We will briefly show here how to derive the equation (2). Consider an electrical RLC circuit as in Figure **??**: it consists of an inductance L, a capacitor C, a resistor R and we assume that the voltage source is a battery ($E(t) = E$ constant). When the battery switch is closed, a current $I(t)$ begins to flow in the circuit according to Kirchhoff's Voltage Law:

$$LI' + RI + \frac{1}{C}Q = E$$

where Q is the charge of the capacitor and $Q' = I$. If we differentiate both sides of the equation, we find $LI'' + RI' + \frac{1}{C}I = 0$, which is a second-order linear ODE with constant coefficients and it represents a damped harmonic oscillator.

The circuit that was considered by Van der Pol, however, displayed an active element (an array of vacuum tubes – semiconductor) instead of a passive resistor. This way, the semiconductor acts as if it is pumping energy in the system, when the current is low, and damping the energy of the system, when the current is too high (unlike a resistor which simply dissipates energy). The interplay between energy injection and energy absorption results in a periodic oscillation in voltages and currents. The action of the semiconductor is modelled by the function $I^2 - \alpha$, where α is the threshold level of current, and the equation for the current becomes

$$LI' + (I^2 - \alpha)I + \frac{1}{C}Q = E$$

where we assumed that there is no external source of voltage, for simplicity. By taking the derivative on both sides of the equation we obtain indeed the VdP equation:

$$LI'' + 3I'\left(I^2 - \frac{\alpha}{3}\right) + \frac{1}{C}I = 0$$

(the constants can be rescaled to have the standard form of the VdP equation (2)).

Since its introduction, the Van der Pol equation has been used as a basic model for oscillatory processes in physics, electronics, biology, neurology, sociology and economics. The model was soon generalized to a forced system

$$y'' + \mu(y^2 - 1)y' + y = F_\omega(t),$$

where $F_\omega(t)$ is an external force, possibly depending on some parameter $\omega \in \mathbb{R}$. In particular, much attention has been devoted to the study of the VdP equation under an external periodic (sinusoidal) force $F_\omega(t) = a\sin(\omega t)$, with $a, \omega \in \mathbb{R}$. Van der Pol himself built a number of electronic circuit models of the human heart to study the range of stability of heart dynamics. His investigations with adding an external driving signal were analogous to the situation in which a real heart is driven by a pacemaker. He was interested in finding out how to stabilize a heart's irregular beating (arrhythmias).

2 Qualitative analysis

The Van der Pol equation

$$y'' + \mu(y^2 - 1)y' + y = 0 \qquad \mu > 0 \tag{3}$$

is a special case of a more general class of second-order non-linear autonomous equations called Liénard equations:

$$y'' + W(y)y' + Z(y) = 0$$

These type of equations can be interpreted as a model for a spring-mass system where the damping force $W(y)$ depends on the position (for example, the mass might be moving through a viscous medium of varying density), and the spring constant $Z(y)$ (or restoring force) depends on how much the spring is stretched: in particular,

$$y'' + \underbrace{\mu(y^2 - 1)}_{\text{damping}}y' + \underbrace{y}_{\text{restoring}} = 0.$$

and the parameter μ in front of the nonlinear term indicates the strength of the damping.

By directly inspecting the equation, we can already draw some general conclusions about the behaviour of the solutions. If the solution $y \gg 1$, both the restoring and damping forces are large, so that $|y(t)|$ should decrease with time. The system behaves like a strongly damped oscillator and it disperses energy.

If the solution is small $|y| \ll 1$, the damping force becomes negative, which should make $|y(t)|$ tend to increase with time. The energy of the system grows.

This is already a hint that there could be a limit cycle in the phase space. We will see that indeed this dynamical system has a unique stable limit cycle or, equivalently, the equation (3) has a unique periodic solution and all nearby solutions tend towards this periodic solution as $t \to +\infty$. Clearly, the "shape" of the limit cycle will strongly depend on the value of μ. Before proving it, let's have a closer look at the equation.

We can rewrite the VdP equation as a first order system with two variables (y, v) (y being the position and v being the velocity):

$$\begin{cases} y' = v \\ v' = \mu(1-y^2)v - y \end{cases} \tag{4}$$

2.1 Equilibrium points

The system admits only one equilibrium point

$$\begin{cases} y' = v = 0 \\ v' = \mu(1-y^2)v - y = 0 \end{cases} \qquad \Leftrightarrow \qquad y = 0,\ v = 0.$$

If we want to analyze the local behaviour, we'll need as usual to linearize the system around $(0, 0)$: the Jacobian of the system at the point $(0, 0)$ is

$$J = \begin{bmatrix} 0 & 1 \\ -1 & \mu \end{bmatrix}$$

with $\operatorname{Tr} J = \mu$ and $\det J = 1$. Its eigenvalues are

$$\lambda_{1,2} = \frac{\mu \pm \sqrt{\mu^2 - 4}}{2}$$

Therefore, depending on the value of μ we may have

$$(0,0) = \begin{cases} \text{center} & \text{if } \mu = 0 \\ \text{unstable spiral point} & \text{if } 0 < \mu < 2 \\ \text{unstable degenerate node} & \text{if } \mu = 2 \\ \text{unstable node} & \text{if } \mu > 2. \end{cases}$$

2.2 Force-lines and symmetries

We want to identify some reference force-lines to start drawing the phase space. We study force-lines that are purely vertical or horizontal in the (y, v) phase plane. In the first case we set

$$y' = 0 \qquad \Leftrightarrow \qquad v = 0$$

meaning that the force-lines become vertical along the y-axis. In the second case, we set

$$v' = 0 \qquad \Leftrightarrow \qquad v = \frac{y}{\mu(1-y^2)}$$

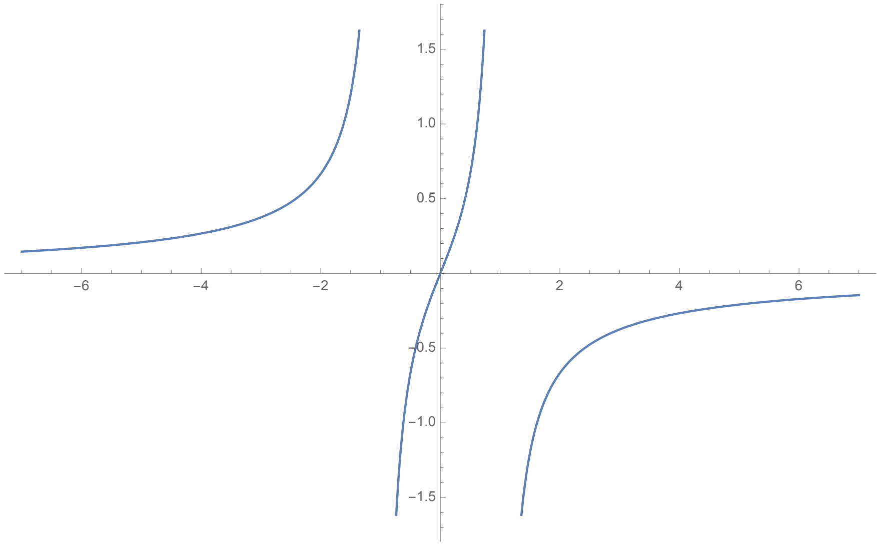

Figure 2: The lines where the force-lines becomes horizontal.

meaning that the force-lines become horizontal along the (3-fold) curve $v = \frac{y}{\mu(1-y^2)}$ as in Figure 2.

It turns out that the system (4) has an additional symmetry, i.e. if the system doesn't change under some given transformation: consider the mapping

$$(y, v) \mapsto (-y, -v)$$

Then, the system (4) becomes

$$\begin{cases} -y' = -v \\ -v' = \mu \left(1 - (-y)^2\right)(-v) - (-y) = -\mu \left(1 - (-y)^2\right) v + y \end{cases}$$

which is exactly the same as (4). This implies that the phase diagram will be symmetric with respect to the origin $(0, 0)$.

Remark 1. *You can also check that by changing the sign of y or v only ($(y, v) \mapsto (-y, v)$, or $(y, v) \mapsto (y, -v)$), the system does change! Therefore these transformations (which represent a reflection with respect to the y- or v-axis) are not symmetries of the system.*

With all the gathered information, we can sketch the orbits on the phase plane: see Figure 3.

3 The case $\mu = 0 \sim$ Harmonic Oscillator

For $\mu = 0$ the equation becomes the classic equation for a harmonic oscillator:

$$y'' + y = 0.$$

We all know this system very well. Its orbits in the phase-plane are the usual circles around the origin.

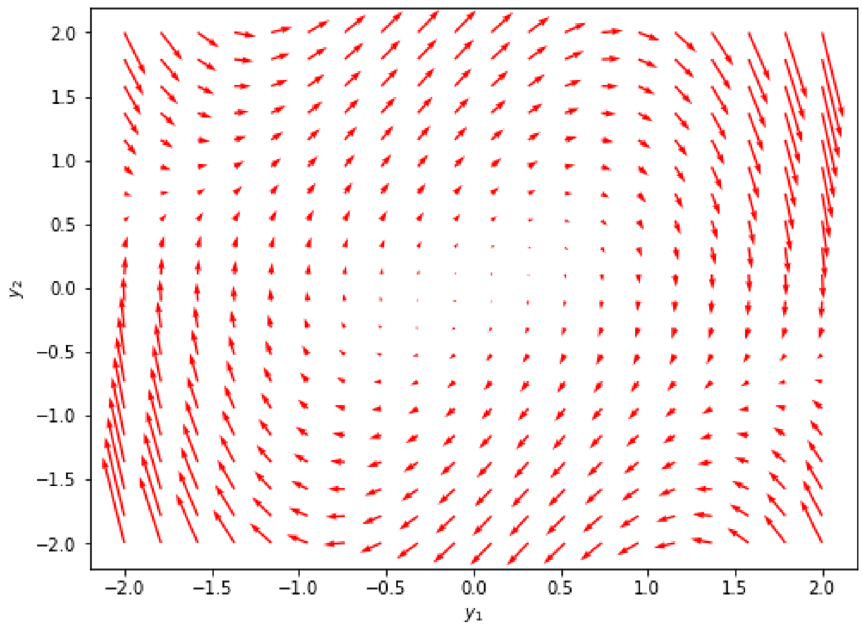

Figure 3: The vector field on the phase-pane.

4 The case $0 < \mu \ll 1 \sim$ Small nonlinearity and the quest for the limit cycle.

4.1 Preliminary discussions

If the nonlinearity is small ($\mu \ll 1$), we can argue that the phase space will be a small distortion of the phase space of the harmonic oscillator. In particular, we may still expect a circular behaviour of the orbits. It is then natural to switch to polar coordinates:

$$\begin{cases} y = r\cos\theta \\ v = r\sin\theta \end{cases} \qquad \Leftrightarrow \qquad \begin{cases} y^2 + v^2 = r^2 \\ \frac{v}{y} = \tan\theta \end{cases}$$

By differentiating

$$rr' = yy' + vv'$$
$$\frac{\theta'}{\cos^2\theta} = \frac{v'y - vy'}{y^2}$$

and substituting (4)

$$rr' = \cancel{yv} - \cancel{vy} + \mu(1-y^2)v^2$$
$$\theta' = \frac{\mu(1-y^2)vy - y^2 - v^2}{r^2},$$

we obtain an equivalent system in the new variables (r, θ):

$$\begin{cases} r' = 0 + \mu r \sin^2\theta(1 - r^2\cos^2\theta) \\ \theta' = -1 + \mu \sin\theta\cos\theta(1 - r^2\cos^2\theta) \end{cases}$$

If $\mu \ll 1$ (say, $\mu \approx 0$), the leading-order terms on the right hand side of the system give the linearized system $r' = 0$ and $\theta' = -1$, meaning that the radius remains constant $r(t) = r_0$ and the angle evolves linearly $\theta(t) = t$. We now continue by considering also the first sub-leading order terms in the systems, i.e. the terms of order $\mathcal{O}(\mu)$, and we consider them as a bounded perturbation of the leading-order system.

We can already make some preliminary observations: if r is small enough, then $r' > 0$ and the orbit is growing, while if r is big enough, then $r' < 0$ and the orbit is decreasing. Also, the angle θ evolves faster than r, since already in leading-order approximation it was evolving linearly (while r remained constant).

4.2 The method of averaging

In the case $\mu \ll 1$, we make an ansatz for the solution:

$$\begin{cases} y(t) = r(t)\cos(t + \omega(t)) \\ y'(t) = -r(t)\sin(t + \omega(t)) \end{cases} \tag{5}$$

the motivation for this ansatz is that when μ is zero equation (2) has its solution of the form (5) with r and ω constants. For small values of μ we expect the same form of the solution to be approximately valid, but now r and ω will be slowly varying functions of t.

From (5), in a similar way as the above calculations, we obtain the system

$$\begin{cases} r' = -\mu r\left(r^2\cos^2(t+\omega) - 1\right)\sin^2(t+\omega) \\ \omega' = -\mu\left(r^2\cos^2(t+\omega) - 1\right)\sin(t+\omega)\cos(t+\omega) \end{cases} \tag{6}$$

The "method of averaging" corresponds to assuming that since the variables (r, ω) are slowly varying in time, they are acting *on average* as constants. In formulæ, we are replacing the right hand side of (6) with their average over one cycle of oscillation (note that the vector-field is 2π periodic):

$$\begin{cases} r' = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} -\mu r\left(r^2\cos^2(t+\omega) - 1\right)\sin^2(t+\omega)\mathrm{d}\omega \\ \omega' = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} -\mu\left(r^2\cos^2(t+\omega) - 1\right)\sin(t+\omega)\cos(t+\omega)\mathrm{d}\omega \end{cases}$$

It remains to calculate the integrals. The equation for ω is equal to zero because we're integrating periodic functions over their period (and r is kept constant):

$$\begin{aligned} &\frac{-\mu}{2\pi}\int_0^{2\pi} r^2\cos^3(t+\omega)\sin(t+\omega) - \sin(t+\omega)\cos(t+\omega)\mathrm{d}\omega \\ &\qquad = \frac{-\mu}{2\pi}\left[-r^2\frac{\cos^4(t+\omega)}{4} + \frac{\cos^2(t+\omega)}{2}\right]_0^{2\pi} = 0 \end{aligned}$$

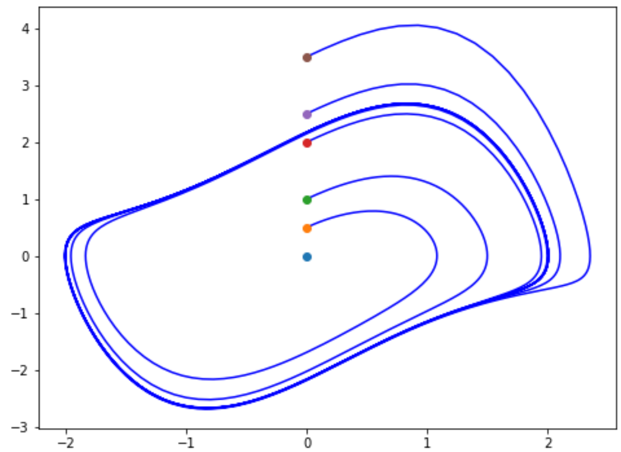

Figure 4: Different trajectories of the system with $\mu = 0.1$.

therefore, $\omega' = 0$ impying $\omega(t) = \omega_0$. For the equation for r we use the trigonometric identities

$$\cos^2(\theta) = \frac{1+\cos(2\theta)}{2} \qquad \sin^2(\theta) = \frac{1-\cos(2\theta)}{2}$$

to obtain

$$\begin{aligned}
r' &= \frac{-\mu r}{2\pi}\int_0^{2\pi} r^2\cos^2(t+\omega)\sin^2(t+\omega) - \sin^2(t+\omega)\mathrm{d}\omega \\
&= \frac{-\mu r}{2\pi}\int_0^{2\pi} r^2\frac{1+\cos(2(t+\omega))}{2}\frac{1-\cos(2(t+\omega))}{2} - \frac{1+\cos(2(t+\omega))}{2}\mathrm{d}\omega \\
&= \frac{-\mu r^3}{2\pi}\int_0^{2\pi}\frac{1-\cos^2(2(t+\omega))}{4}\mathrm{d}\omega + \frac{\mu r}{2} = -\frac{\mu r^3}{4} + \frac{\mu r^3}{4\cdot 2\pi}\int_0^{2\pi}\cos^2(2(t+\omega))\mathrm{d}\omega + \frac{\mu r}{2} \\
&= -\frac{\mu r^3}{4} + \frac{\mu r^3}{4\cdot 2\pi}\int_0^{2\pi}\frac{1+\cos(4(t+\omega))}{2}\mathrm{d}\omega + \frac{\mu r}{2} = -\frac{\mu r^3}{4} + \frac{\mu r^3}{8\pi} + \frac{\mu r}{2} \\
&= -\frac{\mu r^3}{8} + \frac{\mu r}{2} = \frac{\mu r}{8}(4-r^2)
\end{aligned}$$

i.e. a separable equation for r. By partial fraction decomposition we can recover the solution.

First of all, notice that if $r_0 = 0, 2$, then $r(t) = 0$ and $r(t) = 2$ are constant solutions. In particular, $r(t) = 0$ is the equilibrium point of the system (see Section 2.1) and since $\mu \ll 1$, it is an unstable spiral point in the (y, v)-phase plane. The solution $r(t) = 2$ has a negative eigenvalue (linearize the (r, ω)-system near the point $(\omega_0, 2)$ and check its Jacobian...), therefore it is a stable node which corresponds to a stable (almost) 2π-periodic orbit of radius 2.

Assuming now $r(0) = r_0 < 2$ we get

$$8\frac{\mathrm{d}r}{r(4-r^2)} = \mu\mathrm{d}t$$

and by partial fraction decomposition

$$\frac{1}{r(4-r^2)} = \frac{-\frac{1}{4}}{r} + \frac{\frac{1}{8}}{2-r} + \frac{\frac{1}{8}}{2+r}$$

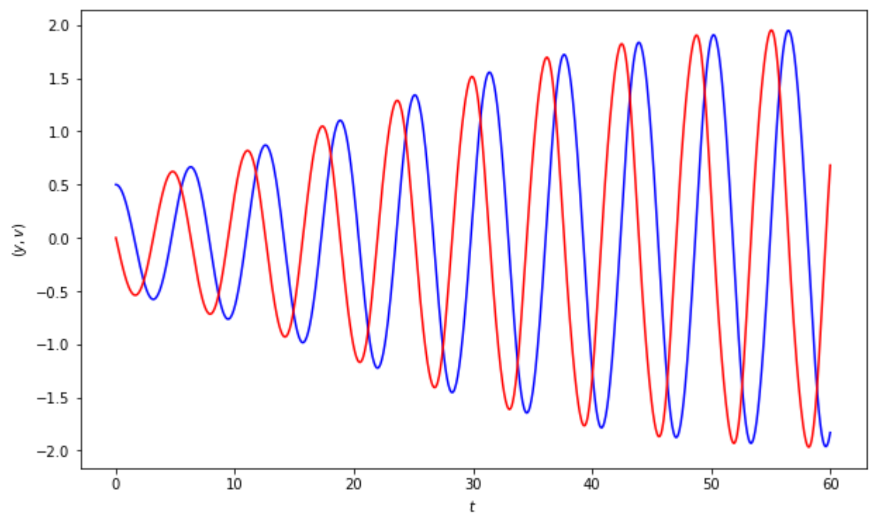

Figure 5: Plot of the solution v (blue) and its derivative v (red) with $\mu = 0.1$ and initial condition $(y_0, v_0) = (0.5, 0)$.

Integrating we get

$$\int 8\frac{\mathrm{d}r}{r(4-r^2)} = 8\left[-\frac{1}{4}\ln r - \frac{1}{8}\ln|2-r| + \frac{1}{8}\ln|2+r|\right] = \ln\left|\frac{r+2}{r^2(r-2)}\right|$$

In conclusion

$$\frac{r+2}{r^2(r-2)} = C_0 e^{\mu t}$$

where $C_0 = \frac{r_0^2(r_0-2)}{r_0+2}$. The above implicit relation can actually be rewritten explicitly

$$r(t) = \frac{2e^{\frac{\mu}{2}t}}{\sqrt{e^{\mu t} - 1 + \frac{4}{r_0^2}}}$$

It is easy to see that as $t \to +\infty$, $r(t) \to 2$ and the limit value is the value of the limit cycle.

5 The case $\mu \gg 1 \sim$ Relaxation oscillations

5.1 Qualitative analysis.

In the case when μ is very big, we can analyze the solutions of the VdP equation in the following way.We start by rewriting the VdP equation 2 as Rewrite the VdP equation as

$$\mu\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{y'}{\mu} + \frac{1}{3}y^3 - y\right] + y = 0$$

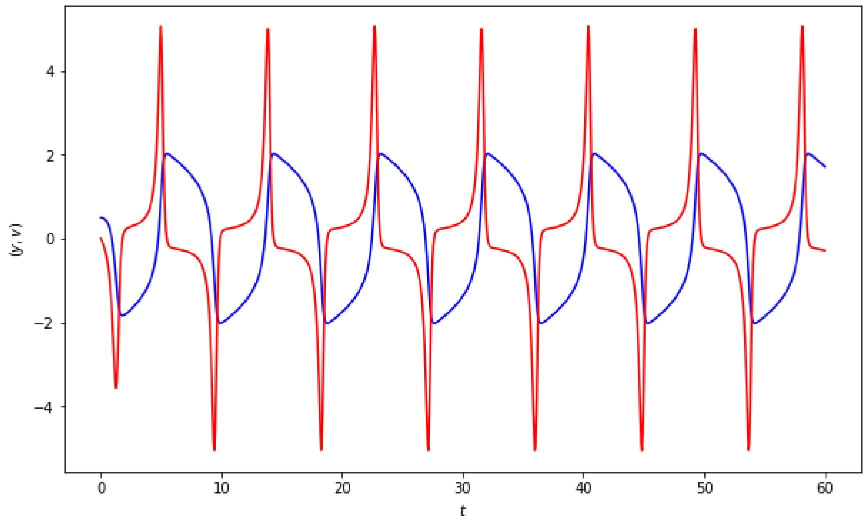

Figure 6: Plot of the solution v (blue) and its derivative v (red) with $\mu = 5$ and initial condition $(y_0, v_0) = (0.5, 0)$.

equivalently,

$$\begin{cases} y' = \mu \left[w - \left(\frac{1}{3} y^3 - y \right) \right] \\ w' = -\frac{y}{\mu} \end{cases}$$

Since $\mu \gg 1$ is very large, we have that $|w'| \leq |y'|$ and actually $w' \approx 0$, meaning that the force lines are horizontal almost everywhere, except in a neighbourhood of the solution $y' = 0$ (a cubic curve Γ given by $w = \frac{y^3}{3} - y$), where the two quantities y' and w' become comparable.

When solutions arrive (horizontally) in a neighbourhood of such cubic Γ in the (y, w)-phase space, they turn sharply and follow Γ until they reach a critical point of Γ:

$$w' = \left(\frac{y^3}{3} - y \right)' = 0 \quad \Leftrightarrow \quad y = \pm 1 \text{ (and } w = \mp \frac{2}{3});$$

when the trajectory is close to one critical points, it leaves Γ and continues again horizontally to the other critical point.

5.2 Existence of the limit cycle

The Poincaré–Bendixson Theorem for the existence of limit cycles cannot be applied in this context, since for $\mu \gg 1$ the force lines are mostly horizontal and we cannot identify a good trapping region. However, the existence of a (stable) limit cycle of the VdP equation for *any* $\mu > 0$ can be proved using a powerful theorem due to Levinson and Smith, which we will not prove.

Theorem 2 (Levinson–Smith Theorem). *Consider the autonomous* 2^{nd} *order differential equation*

$$y'' + f(y)y' + g(y) = 0$$

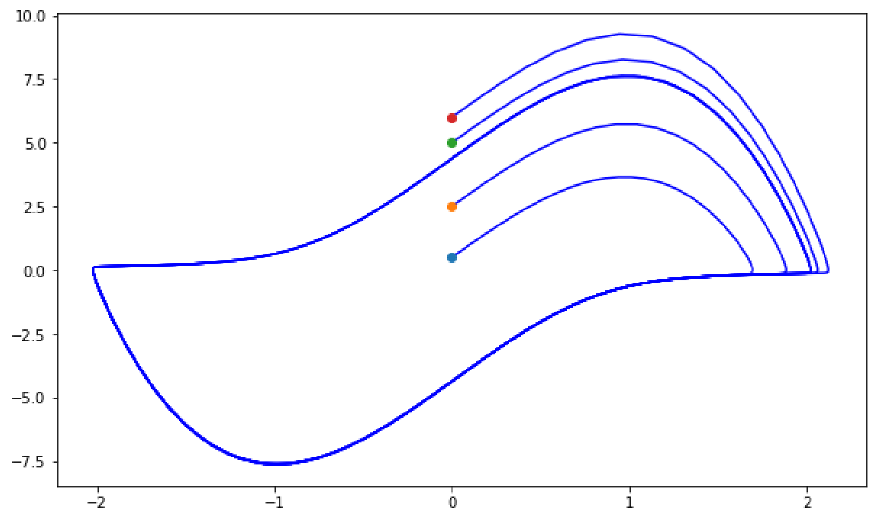

Figure 7: Different trajectories of the system with $\mu = 3$.

and suppose that the following conditions are satisfied:

(a) $f(y)$ *is even and continuous,*

(b) $g(y)$ *is odd,* $g(y) > 0$ *if* $y > 0$*, and* $g(y)$ *is continuous for all* y*,*

(c) $G(y) \to \infty$ *as* $y \to \infty$*, where* $G(y) = \int_0^y g(x)\mathrm{d}x$

(d) for some $k > 0$*, we have that* $F(y) < 0$ *for* $y \in (0, k)$*,* $F(y) > 0$ *and increasing for* $y \in (k, +\infty)$ *and* $F(y) \to \infty$ *as* $y \to \infty$*, where* $F(y) = \int_0^y f(x)\mathrm{d}x$*.*

Then, the corresponding 1^{st} *order system has*

(i) a unique critical point at the origin;

(ii) a unique non-zero closed trajectory Σ*, which is a stable limit cycle around the origin;*

(iii) all other non-zero trajectories are spiralling towards Σ *as* $t \nearrow +\infty$*.*

It is easy to see that the VdP equation satisfies all the conditions (a)–(d) above: $f(y) = \mu(y^2 - 1)$ is a quadratic polynomial, therefore even and continuous on $\mathbb{R}$ and its antiderivative $F(y) = \mu \int_0^y s^2 - 1\,\mathrm{d}s = \mu\left(\frac{y^3}{3} - y\right)$ is negative for $y \in (0, \sqrt{3})$, positive and increasing on $(\sqrt{3}, +\infty)$ and $F(y) \to +\infty$ as $y \to +\infty$; $g(y) = y$ the identity function is an odd function and its antiderivative $G(y) = \int_0^y s\,\mathrm{d}s = \frac{y^2}{2} \to +\infty$ as $y \to \pm\infty$. Therefore, the VdP equation does admit the existence of a unique stable limit cycle.

6 Numerical considerations

Numerically solving the VdP equation is an example of a *stiff problem*: the solution being sought is varying slowly, but there are nearby solutions that vary rapidly, so the numerical method must take small steps to obtain satisfactory results.

Stiffness is an efficiency issue. If we weren't concerned with how much time a computation takes, we wouldn't be concerned about stiffness. Non-stiff methods can solve stiff problems; they just take a long time to do it.

To cure the stiff issue, since you can't change the differential equation or the initial conditions, you'll need to change the numerical method. Methods intended to solve stiff problems efficiently do more work per step, but can take much bigger steps.

This is why, for example, in order to numerically integrate the VdP equation in Matlab, the standard `ode45` doesn't work very well and it is instead used `ode23s` or `ode15s`.