Additional notes on Quadratic forms

Manuela Girotti

MATH 369-05 Linear Algebra I

1 Conics in $\mathbb{R}^2$

Definition 1. A **conic** is a curve in $\mathbb{R}^2$ described by a polynomial of degree 2 in two variables:

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

Theorem 2 (**Reduction Theorem**)**.** *Using a rotation and then a translation we can always reduce any quadratic equation to standard form.*

The standard forms for any possible type of conics are listed below, after the proof.

Sketch of the proof. Given the polynomial

$$0 = ax^2 + bxy + cy^2 + dx + ey + f = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} d & e \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + f$$

consider the quadratic part

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix} Q \begin{bmatrix} x \\ y \end{bmatrix}$$

and apply the Principal Axes Theorem: there exists an (orthogonal) change of variables

$$\begin{cases} x = At + Bs \\ y = Ct + Ds \end{cases}$$

where the matrix of coefficients is $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathcal{O}(2)$ (a rotation) such that

$$\begin{bmatrix} x & y \end{bmatrix} Q \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} t & s \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} t \\ s \end{bmatrix} = \lambda_1 t^2 + \lambda_2 s^2$$

where λ_1, λ_2 are the eigenvalues of Q. Substituting also in the linear part, we will obtain the equation $\lambda_1 t^2 + \lambda_2 s^2 + d\,(At + Bs) + e\,(Ct + Ds) + f = 0$, meaning

$$\lambda_1 t^2 + \lambda_2 s^2 + (Ad + Ce)\,t + (Bd + De)\,st + f = 0.$$

At this point we can complete the squares and apply a new change of variables of the form

$$\begin{bmatrix} t \\ s \end{bmatrix} = \begin{bmatrix} \xi \\ \eta \end{bmatrix} + \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

where $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ a fixed vector to be determined by completing the square (this change of variables represents a translation); in the end we reduce the equation to a purely quadratic one (plus a constant):

$$\lambda_1 \xi^2 + \lambda_2 \eta^2 = k.$$

$\square$

Important consequence:
The type of conic that we can obtain depends from the eigenvalues of Q!

1.1 Classification of Conics. Standard forms

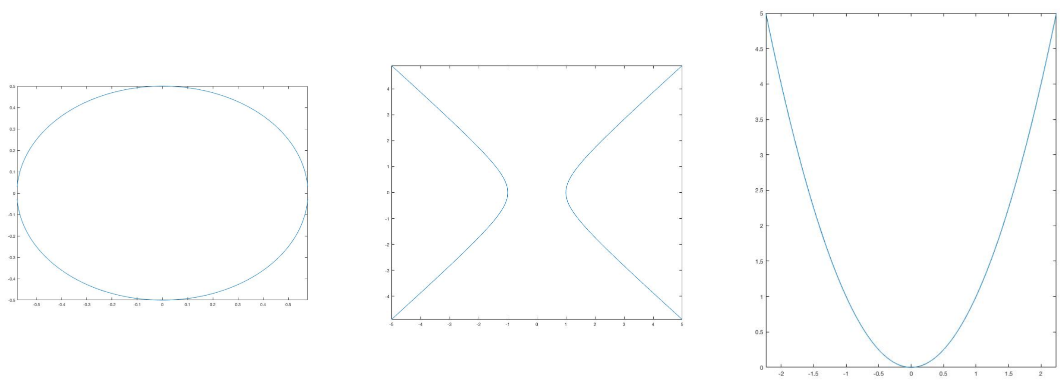


Figure 1: Ellipses, Hyperbola and Parabola

We know that $\det(Q) = \lambda_1 \lambda_2$ and thanks to the Reduction Theorem every conic can be reduced into **standard form**

$$\lambda_1 x^2 + \lambda_2 y^2 = k.$$

Without loss of generality, the constant k can be either zero $k = 0$ or it can be non-zero and therefore normalize it to $k = 1$.

- if $\det(Q) > 0$, then λ_1 and λ_2 have the same sign. Therefore,
  - if $k = 1$ and $\lambda_1, \lambda_2 > 0$, the conic we have is an ellipse: $\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1$.
  - if $k = 1$ and $\lambda_1, \lambda_2 < 0$, we have an imaginary ellipse: $-\frac{x^2}{\alpha^2} - \frac{y^2}{\beta^2} = 1$
  - if $k = 0$, we have two complex conjugate lines meeting in one real point: $\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = \left(\frac{x}{\alpha} + i\frac{y}{\beta}\right)\left(\frac{x}{\alpha} - i\frac{y}{\beta}\right) = 0$ (the set of real solutions is just one point $(0, 0)$)
- if $\det(Q) < 0$, then λ_1 and λ_2 have opposite sign. Therefore,
  - if $k \neq 0$, we have a hyperbola: $\frac{x^2}{\alpha^2} - \frac{y^2}{\beta^2} = 1$ or $-\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1$

- if $k = 0$ we have a pair of real lines meeting in one point: $\frac{x^2}{\alpha^2} - \frac{y^2}{\beta^2} = \left(\frac{x}{\alpha} + \frac{y}{\beta}\right)\left(\frac{x}{\alpha} - \frac{y}{\beta}\right) = 0$

- if $\det(Q) = 0$, then one of the eigenvalue is zero, say $\lambda_2 = 0$ and $\lambda_1 \neq 0$ (they cannot be both zero, otherwise we don't have a conic to begin with). In this case we can complete the square only for the first variable and we are left with something like

$$\lambda_1 x^2 + hs = k$$

for some constants h, k. Now,

  - if $h \neq 0$, we can further reduce it to a parabola: $\lambda_1 x^2 + y = 0$.
  - if $h = 0$, then we have the equation

$$x^2 = \frac{k}{\lambda_1}$$

and, according to the sign of $\frac{k}{\lambda_1}$, we have:

(a) two parallel real lines for $\frac{k}{\lambda_1} > 0$: $x = \pm\sqrt{\frac{k}{\lambda_1}}$

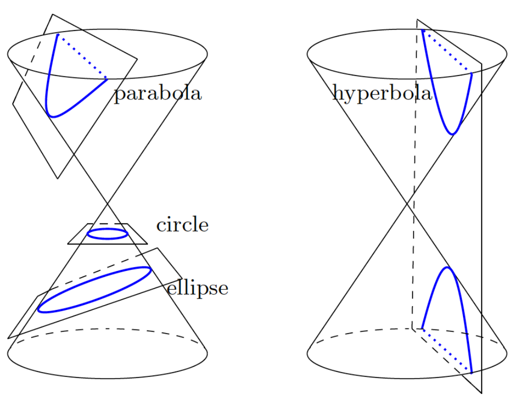
(b) one double line for $\frac{k}{\lambda_1} = 0$: $x^2 = 0$

(c) two parallel complex conjugate lines for $\frac{k}{\lambda_1} < 0$: $x = \pm i\sqrt{\frac{k}{\lambda_1}}$

Conclusion: these are all the possible conics (up to congruence and similarity). There are no others types of conics in the plane.

Why "conics"? They are called conics because they are all slices of the $3D$ cone

$$x^2 + y^2 = z^2.$$



2 Quadric surfaces in $\mathbb{R}^3$

Definition 3. A **quadric surface** is a surface in $\mathbb{R}^3$ described by a polynomial of degree 2 in three variables:

$$ax^2 + by^2 + cz^2 + dxy + exz + fyz + gx + hy + jz + k = 0$$

or in matrix form

$$\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} a & \frac{d}{2} & \frac{e}{2} \\ \frac{d}{2} & b & \frac{f}{2} \\ \frac{e}{2} & \frac{f}{2} & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} f & g & j \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + k = \begin{bmatrix} x & y & z \end{bmatrix} Q \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} f & g & j \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + k = 0$$

The Reduction Theorem is still valid with the rotation matrix $P \in \mathcal{O}(3)$ (plus, eventually, a translation). Therefore, every quadric surface in $\mathbb{R}^3$ can be reduced to a standard form (as listed below). If all the eigenvalues are different from zero, the standard form looks like

$$\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 = \kappa$$

where $\lambda_1, \lambda_2, \lambda_3$ are the eigenvalues of Q, for some constant κ (that can be taken equal to either 0 or 1 without loss of generality). If one or more eigenvalues are zero, we need to be careful when applying the Reduction Theorem (as we did in $\mathbb{R}^2$ when $\det(Q) = 0$).

2.1 Classification of Quadric Surfaces. Standard forms

Important consequence: The type of quadric surface is again determined by the eigenvalues of the matrix Q, more specifically by the sign of the eigenvalues or the vanishing of one or two eigenvalues.

Definition 4. The **signature** of a real symmetric matrix Q is the triplet (p, q, r) that indicates the number (counted with multiplicity) of positive, negative and zero eigenvalues of Q.

Depending on the signature we have:

- $(3, 0, 0)$ (3 positive eigenvalues, 0 negative eigenvalues and 0 0-eigenvalue)
  - ellipsoid: $\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} + \frac{z^2}{\gamma^2} = 1$
  - imaginary cone with vertex at the origin: $\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} + \frac{z^2}{\gamma^2} = 0$
- $(2, 1, 0)$ (2 positive eigenvalues, 1 negative eigenvalue and 0 0-eigenvalue)
  - hyperboloid of one sheet: $\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} - \frac{z^2}{\gamma^2} = 1$
  - cone with vertex at the origin: $\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} - \frac{z^2}{\gamma^2} = 0$
- $(1, 2, 0)$ (1 positive eigenvalue, 2 negative eigenvalues and 0 0-eigenvalue)
  - hyperboloid of two sheets: $\frac{x^2}{\alpha^2} - \frac{y^2}{\beta^2} - \frac{z^2}{\gamma^2} = 1$
- $(0, 3, 0)$ (0 positive eigenvalues, 3 negative eigenvalues and 0 0-eigenvalue)

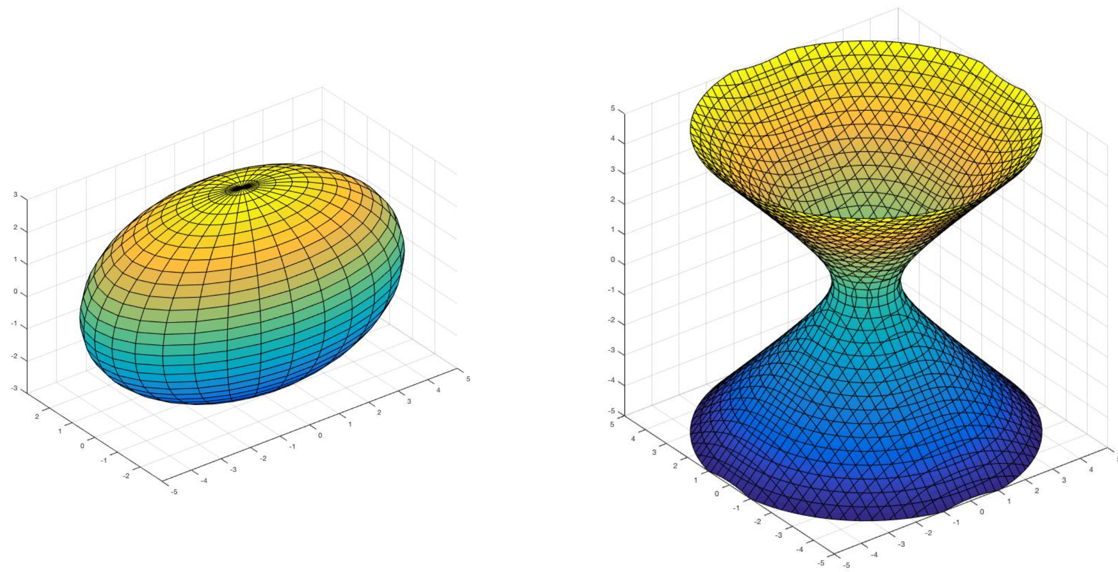


Figure 2: Ellipsoid (left) and Hyperboloid of 1 sheet (right)

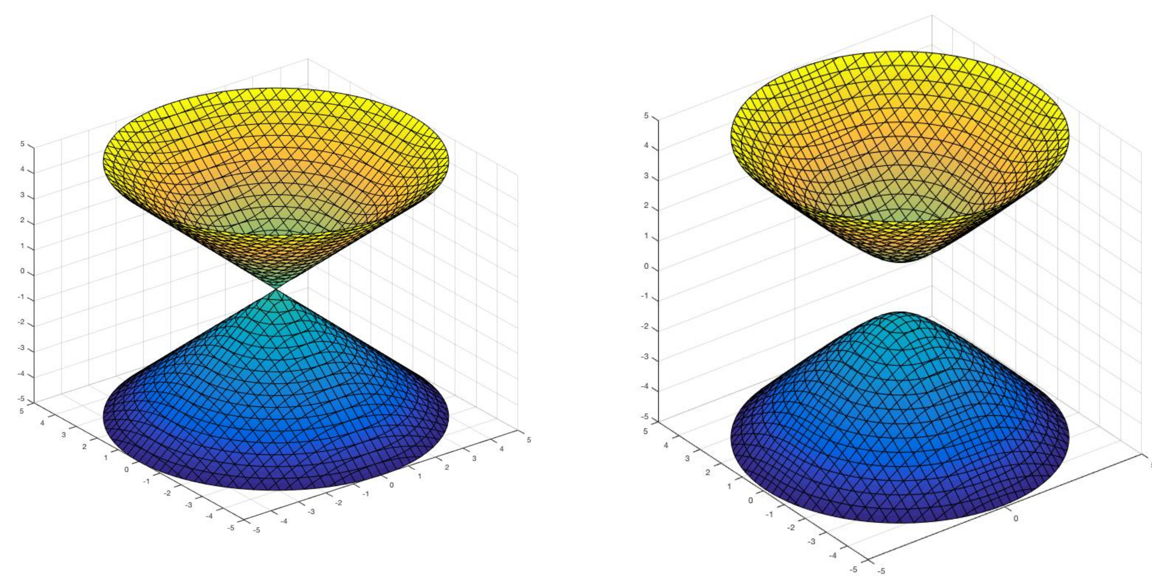


Figure 3: Cone (left) and Hyperboloid of 2 sheets (right)

- imaginary ellipsoid: $-\frac{x^2}{\alpha^2} - \frac{y^2}{\beta^2} - \frac{z^2}{\gamma^2} = 1$

- $(2,0,1)$ (2 positive eigenvalues, 0 negative eigenvalues and 1 0-eigenvalue)

  - elliptic paraboloid: $z = \frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2}$
  - elliptic cilinder: $\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1$

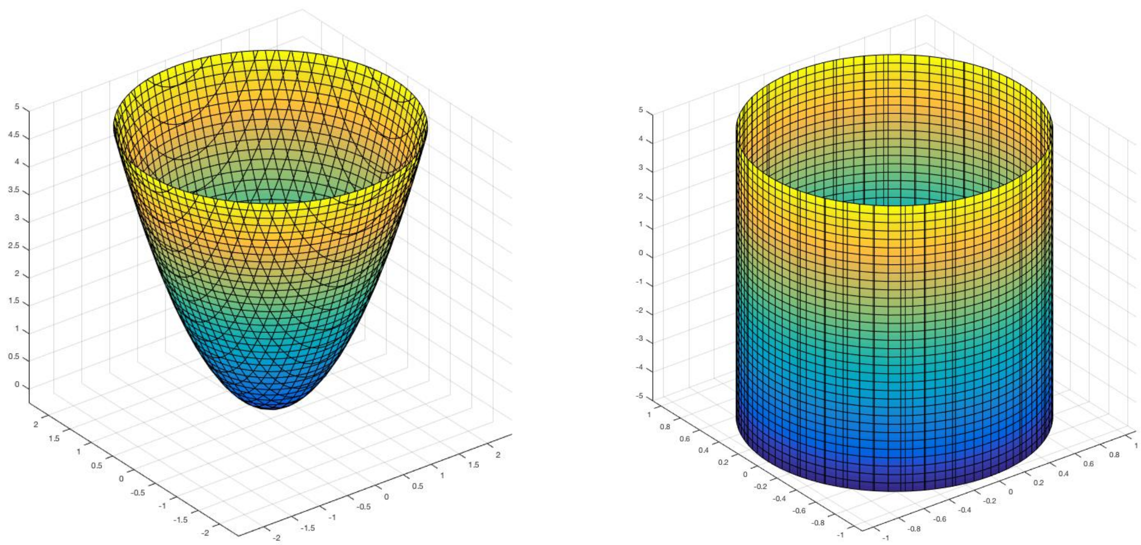


Figure 4: Elliptic paraboloid (left) and Elliptic cylinder (right)

  - two complex conjugate planes containing the z-axis: $\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = \left(\frac{x}{\alpha} + i\frac{y}{\beta}\right)\left(\frac{x}{\alpha} - i\frac{y}{\beta}\right) = 0$ (the set of real solutions is the line $(0,0,t)$)

- $(1,1,1)$ (1 positive eigenvalue, 1 negative eigenvalue and 1 0-eigenvalue)

  - hyperbolic paraboloid: $z = \frac{x^2}{\alpha^2} - \frac{y^2}{\beta^2}$
  - hyperbolic cilinder: $\frac{x^2}{\alpha^2} - \frac{y^2}{\beta^2} = 1$
  - two planes containing the z-axis: $\frac{x^2}{\alpha^2} - \frac{y^2}{\beta^2} = \left(\frac{x}{\alpha} + \frac{y}{\beta}\right)\left(\frac{x}{\alpha} - \frac{y}{\beta}\right) = 0$

- $(0,2,1)$ (0 positive eigenvalues, 2 negative eigenvalues and 1 0-eigenvalue)

  - imaginary cilinder: $-\frac{x^2}{\alpha^2} - \frac{y^2}{\beta^2} = 1$

- $(1,0,2)$ and $(0,1,2)$ (1 positive eigenvalue, 0 negative eigenvalue or 0 positive eigenvalues, 1 negative eigenvalue, and 2 0-eigenvalues)

  - parabolic cilinder: $y = \alpha x^2 + \beta$
  - two parallel planes: $\frac{x^2}{\alpha^2} = 1 \Rightarrow x = \pm\alpha$

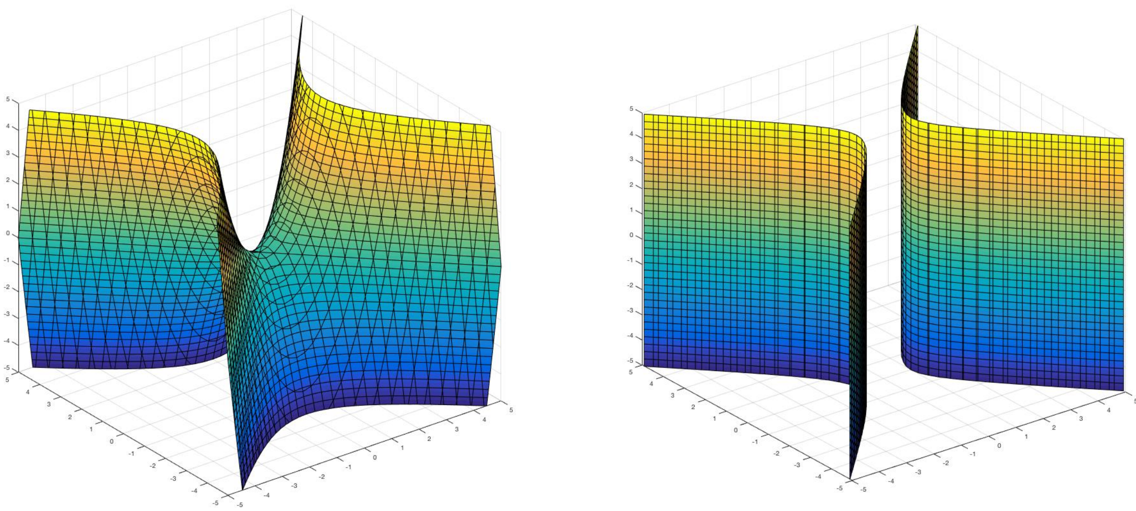


Figure 5: Hyperbolic paraboloid (left) and Hyperbolic cylinder (right)

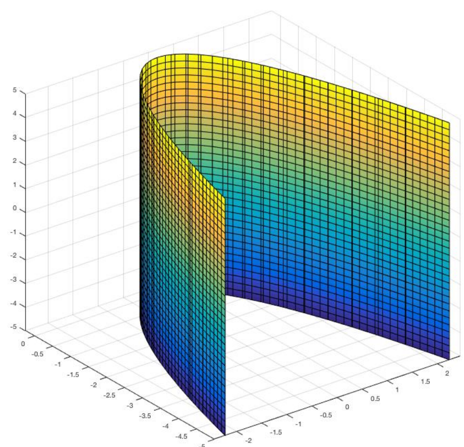


Figure 6: Parabolic cylinder

- two coincident planes: $x^2 = 0$
- two parallel complex conjugate planes: $-\frac{x^2}{\alpha^2} = 1 \Rightarrow x = \pm\alpha i$

Conclusion: these are all the possible quadric surfaces (up to congruence and similarity). There are no others types of quadrics in the $3D$-space.

Note 5. *The classification of quadric hypersurfaces in $\mathbb{R}^n$ for general n follows the same criteria (study of eigenvalues of the quadric and their mutual sign and -possible- vanishing).*

3 How to identify a conic in $\mathbb{R}^2$? Example.

Step 0. You have the conic

$$3x^2 - 2xy + 3y^2 - 8\sqrt{2}x + 10 = 0$$

and you rewrite it in matrix form

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -8\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + 10 = 0$$

Step 1 - Eigenvalues. Consider only the quadratic part, in particular the matrix

$$Q = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$$

and study its eigenvalues:

$$\det(\lambda I - Q) = (\lambda - 3)^2 - 1 = \lambda^2 - 6\lambda + 8 = 0$$

The eigenvalues are: $\lambda_1 = 2 > 0$ and $\lambda_2 = 4 > 0$ (*guess: this could be an ellipse or* 2 *complex lines intersecting in one point*).

Step 2 - Eigenvectors. Orthogonally diagonalizes Q. First of all we look for a basis for each eigenspace W_λ.

$\lambda = 2$**:**

$$(2I - Q)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

meaning solving the system

$$\begin{cases} -x + y = 0 \\ -x + y = 0 \end{cases}$$

The solutions are

$$W_2 = \left\{ \begin{bmatrix} x \\ x \end{bmatrix},\ x \in \mathbb{R} \right\} = \left\langle \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\rangle$$

$\lambda = 4$**:**

$$(4I - Q)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

meaning solving the system

$$\begin{cases} x + y = 0 \\ x + y = 0 \end{cases}$$

The solutions are

$$W_4 = \left\{ \begin{bmatrix} -y \\ y \end{bmatrix},\ y \in \mathbb{R} \right\} = \left\langle \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\rangle$$

Step 3 - Gram-Schmidt. Apply Gram-Schmidt procedure to find an orthonormal basis for R^2. The vectors $\begin{bmatrix}1\\1\end{bmatrix}$ and $\begin{bmatrix}-1\\1\end{bmatrix}$ are already mutually orthogonal, so we just need to normalize them.

$$\mathcal{B} = \left\{ \vec{b}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix}1\\1\end{bmatrix}, \vec{b}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix}-1\\1\end{bmatrix} \right\}$$

Step 4 - Orthogonal matrix P and orthogonal change of variables. Construct the matrix

$$P = \left[\ \vec{b}_1 \mid \vec{b}_2\ \right] = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

P is a 2×2 orthogonal matrix such that

$$P^T Q P = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}.$$

Apply the change of variables

$$\begin{bmatrix}x\\y\end{bmatrix} = P \begin{bmatrix}t\\s\end{bmatrix} \quad \text{meaning} \quad \begin{cases} x = \frac{1}{\sqrt{2}}t - \frac{1}{\sqrt{2}}s \\ y = \frac{1}{\sqrt{2}}t + \frac{1}{\sqrt{2}}s \end{cases}$$

to the conic. This way, we get rid of the cross product in the quadratic part:

$$\begin{bmatrix}x & y\end{bmatrix} Q \begin{bmatrix}x\\y\end{bmatrix} = \left(P\begin{bmatrix}t\\s\end{bmatrix}\right)^T Q \left(P\begin{bmatrix}t\\s\end{bmatrix}\right) = \begin{bmatrix}t & s\end{bmatrix} P^T Q P \begin{bmatrix}t\\s\end{bmatrix} = \begin{bmatrix}t & s\end{bmatrix} \begin{bmatrix}2 & 0\\0 & 4\end{bmatrix} \begin{bmatrix}t\\s\end{bmatrix}$$

Applying this change of variables to the whole conic will give us:

$$\begin{aligned}
0 &= 3x^2 - 2xy + 3y^2 - 8\sqrt{2}x + 10 \\
&= 3\left(\frac{1}{\sqrt{2}}t - \frac{1}{\sqrt{2}}s\right)^2 - 2\left(\frac{1}{\sqrt{2}}t - \frac{1}{\sqrt{2}}s\right)\left(\frac{1}{\sqrt{2}}t + \frac{1}{\sqrt{2}}s\right) + 3\left(\frac{1}{\sqrt{2}}t + \frac{1}{\sqrt{2}}s\right)^2 - 8\sqrt{2}\left(\frac{1}{\sqrt{2}}t - \frac{1}{\sqrt{2}}s\right) + 10 \\
&= \ldots \text{expand and simplify} \ldots \\
&= 2t^2 + 4s^2 - 8t + 8s + 10
\end{aligned}$$

Step 5 - Complete the square. Complete the square for the variables t and s:

$$\begin{aligned}
0 &= 2t^2 + 4s^2 - 8t + 8s + 10 = 2\left(t^2 - 4t + 4\right) + 4\left(s^2 + 2s + 1\right) - 8 - 4 + 10 \\
&= 2(t-2)^2 + 4(s+1)^2 - 2
\end{aligned}$$

and obtain

$$2(t-2)^2 + 4(s+1)^2 = 2.$$

This is an ellipse centered at the point $(2, -1)$ with semi-axes $\alpha = \frac{1}{\sqrt{2}}$ and $\beta = \frac{1}{2}$.

Note 6. *The final change of variables (the translation) that will transform the conic into standard form will be*

$$\begin{cases} t = \xi + 2 \\ s = \eta - 1 \end{cases}$$

Substituting this expression into the conic above, we get (dividing by 2*)*

$$\xi^2 + 2\eta^2 = 1.$$