1. Describe the following relations by listing the elements of the relation:

   (a) Let $A = \{1, 2, 3, 4, 5\}$ and $R_<$ is the relation "strictly less than".

   (b) Let $A = \{1, 2, 3, 4, 5\}$ and $R_|$ is the relation "divides".

2. Explain with details if

   (a) $R_<$ is reflexive symmetric antisymmetric transitive

   (b) $R_|$ is reflexive symmetric antisymmetric transitive

   (c) Consider the relation $R = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$.
   R is reflexive symmetric antisymmetric transitive

   (d) Consider the relation $R_\leq = \{(x,y) \in \mathbb{R}^2 \mid x \leq y\}$.
   $R_\leq$ is reflexive symmetric antisymmetric transitive

(e) Let $A = \{1, 2, 3, 4\}$, and $R = \{(1,1), (2,2), (2,4), (3,2), (3,3), (4,3), (4,4)\}$. Draw a graph with whose vertices are the elements of A. Draw an arrow from a to b if aRb.
Is this relation reflexive? Symmetric? Transitive?

(f) Let $A = \{3, 5, 6, 9, 30\}$ rand let R be the relation "be multiple of". Draw the arrow diagram of the relation.

(g) Let $A = \mathbb{Z}_+$ and let R be the relation xRy if $x + y$ is odd. Verify if the relation R is an equivalence relation.

(h) Let $A = \mathbb{Z}_+$ and let R be the relation "have the same unit digit". Verify that it is an equivalence relation and study its equivalence classes:

i. how many equivalence classes are there?

ii. how many elements are there in each equivalence class?

(i) Prove the following:

Proposition Let R be an equivalence relation on a set A. Suppose $a, b \in A$. Then $[a] = [b]$ if and only if aRb.

Modular Arithmetic

Definition Given a natural number $n \neq 1$, we define a relation $\equiv$ on $\mathbb{Z}$, called **congruence** mod n, by $a \equiv b$ if $n \mid (a-b)$.

Note: $n \mid a-b$ is equivalent to the following statements:

(a) $n \mid a-b$.

(b) $a = nh + b$ for some $h \in \mathbb{Z}$.

(c) $b = nk + a$ for some $k \in \mathbb{Z}$.

Therefore, two integers a and b are congruent modulo n if they give the same remainder upon division by n.

Examples

$$\begin{aligned} 8 &\equiv 5 \mod 3 \\ 11 &\equiv 5 \mod 3 \\ 2 &\equiv 5 \mod 3 \\ 14 &\equiv 5 \mod 3 \\ -1 &\equiv 5 \mod 3 \\ 28 &\equiv 4 \mod 2 \\ 5678 &\equiv 0 \mod 2 \\ 17 &\equiv 1 \mod 2 \end{aligned}$$

(j) Let $n \in \mathbb{N}$. Prove that the relation $\equiv \mod n$ on the set $\mathbb{Z}$ is an equivalence relation.

(k) Let $A = \mathbb{Z}$, and $R =$ congruence $\mod 7$. Describe the equivalence classes.

(l) Demonstrate that $7 \equiv 1 \mod 6$ and $57 \equiv -13 \mod 7$.

(m) Express "x is even" and "x is odd" in terms of congruences.

(n) What does $x \equiv 0 \mod n$ means in terms of divisibility?

Calculations with Modular Arithmetic

One can work with modular equations in many of the ways as with ordinary equations.

Proposition Let $n \in \mathbb{Z}$.

(a) If $a = b \mod n$ and $c = d \mod n$, then $a + c = b + d \mod n$.

(b) If $a = b \mod n$ and $c = d \mod n$, then $ac = bd \mod n$.

Examples

$$\begin{aligned}
12 + 9 \equiv 21 \equiv 1 &\mod 5\,, \text{ but also } 12 + 9 \equiv 2 + (-1) \equiv 1 \mod 5\\
12 - 9 \equiv 3 &\mod 5\,, \text{ but also } 12 - 9 \equiv 2 - (-1) \equiv 3 \mod 5\\
12 + 3 \equiv 15 \equiv 0 &\mod 5\,, \text{ but also } 12 + 3 \equiv 2 + 3 \equiv 5 \equiv 0 \mod 5\\
15 - 23 \equiv -8 \equiv 2 &\mod 5\,, \text{ but also } 15 - 23 \equiv 0 - (-2) \equiv 2 \mod 5\\
35 \cdot 7 \equiv 245 \equiv 0 &\mod 5\,, \text{ but also } 35 \cdot 7 \equiv 0 \cdot 2 \equiv 0 \mod 5\\
-47 \cdot (5 + 1) \equiv -282 \equiv 3 &\mod 5\,, \text{ but also } -47 \cdot (5 + 1) \equiv 2 \cdot 1 \equiv 2 \mod 5\\
37^3 \equiv 50653 \equiv 3 &\mod 5\,, \text{ but also } 37^3 \equiv 2^3 \equiv 8 \equiv 3 \mod 5
\end{aligned}$$

(o) Prove the proposition above.

(p) Prove that for all integers $n \in \mathbb{Z}$, $n^2 \equiv 0 \mod 4$ or $n^2 \equiv 1 \mod 4$.

(q) Solve $3x + 4 = 2x + 8 \mod 9$.