Partitions

Example Let $A = \{1, 2, 3, 4\}$. Consider each of the following:

- $\{\{1,2\}, \{3,4\}\}$
- $\{\{1\}, \{2\}, \{3\}, \{4\}\}$
- $\{\{1,2,3\}, \{4\}\}$
- $\{\{1,2,3,4\}\}$

Example Let $A = \mathbb{Z}$ and consider the collection of subsets:

$$\{\{\ldots, -4, -2, 0, 2, 4, \ldots\}, \{\ldots, -3, -1, 1, 3, 5, \ldots\}\}$$

Non-examples Let $A = \{1, 2, 3, 4\}$. Consider each of the following:

- $\{\{1,2\}, \{1,3,4\}\}$
- $\{\{1\}, \{2\}, \{3\}, \{4\}, \{\emptyset\}\}$
- $\{\{1\}, \{2\}, \{3\}, \{4\}, \emptyset\}$

1. Which of the following collections of subsets of the plane $\mathbb{R}^2$ are partitions? Explain why.

   (a) $\{\{(x,y) \mid x + y = c\} \mid c \in \mathbb{R}\}$

   (b) The set of all circles in $\mathbb{R}^2$

   (c) The set of all circles in $\mathbb{R}^2$ centered at the origin together with the set $\{(0,0)\}$

   (d) $\{\{(x,y)\} \mid (x,y) \in \mathbb{R}^2\}$

The Pigeonhole Principle

2. Given any 11 integers, there is a pair of numbers whose difference is divisible by 10.

3. Given 5 points in a unit square, there are two points within $\frac{\sqrt{2}}{2}$ of each other.

4. Consider the first $2n$ integers, 1,2,...,$2n$. If more than n of these integers are selected, show that two of the selected integers must be relatively prime.

5. Suppose a soccer team scores at least one goal in 20 consecutive games. If it scores a total of 30 goals in those 20 games, prove that in some sequence of consecutive games it scores exactly 9 goals.

6. Suppose n baseball teams play in a tournament ($n \geq 2$). If each team meets every other team at most once, prove that two teams played the same number of games.

7. Is it possible to cover a chessboard with dominoes that each cover exactly two squares? What if the squares in two diagonally opposite corners are removed?