Partitions of integers

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MATH 250 Fundamentals of Math

Definition 1. If n is a positive integer, then a **partition of** n is a non-increasing sequence of positive integers $p_1, p_2, \ldots, p_k$ whose sum is n, i.e.

$$0 < p_1 \leq p_2 \leq \ldots \leq p_k \,, \qquad \text{such that } \sum_{i=1}^{k} p_i = n \,.$$

A partition is usually denoted with the Greek letter

$$\lambda = (p_k, p_{k-1}, \ldots, p_2, p_1) \,.$$

Each number p_i is called a **part** of the partition. We let the function $p(n)$ denote the number of partitions of the integer n.

As an example, $p(5) = 7$, and here are all 7 of the partitions of the integer $n = 5$:

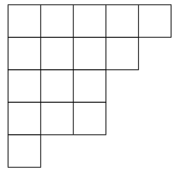
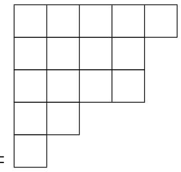
$$\begin{aligned} 5 &= 5 \\ &= 4 + 1 \\ &= 3 + 2 \\ &= 3 + 1 + 1 \\ &= 2 + 2 + 1 \\ &= 2 + 1 + 1 + 1 \\ &= 1 + 1 + 1 + 1 + 1 \end{aligned}$$

We take $p(n) = 0$ for all negative values of n and $p(0)$ is defined to be 1.

Integer partitions were first studied by Euler. For many years one of the most intriguing and difficult questions about them was determining the asymptotic properties of $p(n)$ as n gets large. This question was finally answered quite completely by three mathematicians: Hardy, Ramanujan, and Rademacher.

An example of a problem in the theory of integer partitions that remains unsolved, despite a good deal of effort having been expended on it, is to find a simple criterion for deciding whether $p(n)$ is even or odd. Though values of $p(n)$ have been computed for n into the billions, no pattern has been discovered to date.

A useful way to visualize an integer partition, is the Ferrers diagram: it is constructed by stacking left-justified rows of cells, where the number of cells in each row corresponds to the size of a part. The first row corresponds to the largest part, the second row corresponds to the second largest part, and so on. See figure below for an example of a partition of $n = 16$: $\lambda = (5, 4, 3, 3, 1)$.

$\lambda =$  $\lambda^* =$ 

Given a partition λ, we can create the **conjugate partition** λ^*, by exchanging the rows and the column (see again figure above: $\lambda^* = (5, 4, 4, 2, 1)$). In particular, a partition is called **self-conjugate** if $\lambda^* = \lambda$: in this case the Ferrers diagram is symmetric (for example, the partition $\lambda = (5, 5, 3, 2, 2)$).

As an example of the use of Ferrers diagrams in partition theory, we prove the following.

Theorem 2. *The number of partitions of the integer n whose largest part is k is equal to the number of partitions of n with k parts.*

Proof. To prove this theorem we stare at a Ferrers diagram of the partition λ and notice that if we interchange the rows and columns (i.e. we construct its conjugate λ^*) we have a 1-1 correspondence between the two kinds of partitions. □

Getting back to calculating the number of partitions of a given integer n, there is no simple formula for $p(n)$, but it is not hard to find a generating function for them. A generating function is a formal power series whose coefficients encode the elements of a sequence. In our case, the generating function of $\{p(n)\}_{n=0}^{\infty}$ is

$$\sum_{n=0}^{\infty} p(n)x^n \,.$$

Consider the following product:

$$(1 + x + x^2 + x^3 + \ldots)(1 + x^2 + x^4 + x^6 + \ldots)(1 + x^3 + x^6 + \ldots)(1 + x^4 + x^8 + \ldots) = \prod_{k=1}^{\infty} \sum_{i=0}^{\infty} x^{ik}$$

We claim that by expanding this product, we obtain exactly the generating function we're looking for, namely $\sum_{n=0}^{\infty} p(n)x^n$.

To illustrate, consider the coefficient of x^3. By choosing x from the first parenthesis, x^2 from the second, and 1 from the remaining parentheses, we obtain a contribution of 1 to the coefficient of x^3. Similarly, if we choose x^3 from the third parenthesis, and 1 from all others, we will obtain another contribution of 1 to the coefficient of x^3. The only three ways to obtain x^3 from this infinite product are the following:

$$\begin{aligned}
1 \cdot 1 \cdot x^3 \cdot 1 \ldots \quad &= \{1 \text{ from 1st parenthesis (powers of } x)\} \times \{\ 1 \text{ from 2nd parenthesis (powers of } x^2)\ \} \\
&\quad \times \{\ x^3 \text{ from the 3rd parenthesis (powers of } x^3)\ \} \\
&\quad \times \{1 \text{ from 4th parenthesis (powers of } x^4)\} \ldots \\
x \cdot x^2 \cdot 1 \cdot 1 \ldots \quad &= \{x \text{ from 1st parenthesis (powers of } x)\} \times \{\ x^2 \text{ from 2nd parenthesis (powers of } x^2)\ \} \\
&\quad \times \{\ 1 \text{ from the 3rd parenthesis (powers of } x^3)\ \} \\
&\quad \times \{1 \text{ from 4th parenthesis (powers of } x^4)\} \ldots \\
x^2 \cdot x \cdot 1 \cdot 1 \ldots \quad &= \{x^2 \text{ from 1st parenthesis (powers of } x)\} \times \{\ x \text{ from 2nd parenthesis (powers of } x^2)\ \} \\
&\quad \times \{\ 1 \text{ from the 3rd parenthesis (powers of } x^3)\ \} \\
&\quad \times \{1 \text{ from 4th parenthesis (powers of } x^4)\} \ldots
\end{aligned}$$

So how does this relate to integer partitions?

Let the monomial chosen from the i-th parenthesis $1 + x^i + x^{2i} + x^{3i} \dots$ in the formula above represent the number of times the part i appears in the partition. In particular, if we choose the monomial $x^{c_i i}$ from the i-th parenthesis, then the value i will appear c_i times in the partition. Each selection of monomials makes one contribution to the coefficient of x^n and in general, each contribution must be of the form

$$x^{1c_1} \cdot x^{2c_2} \cdot x^{3c_3} \dots = x^{c_1+2c_2+3c_3\dots}.$$

Thus the coefficient of x^n is the number of ways of writing

$$n = c_1 + 2c_2 + 3c_3 + \dots$$

where each $c_i \geq 0$. Notice that this is just another way to represent an integer partition. For example, the partition $25 = 6+4+4+3+2+2+2+1+1$ could be represented by $25 = 6(1) + 5(0) + 4(2) + 3(1) + 2(3) + 1(2)$ (each partition is uniquely described by the number of 1s, number of 2s, and so on).
Thus, there is a $1-1$ correspondence between choosing monomials whose product is x^n out of the parentheses and the partitions of the integer n.

Now return to the original product

$$\prod_{k=1}^{\infty}\sum_{i=0}^{\infty} x^{ik} = (1 + x + x^2 + x^3 + \dots)(1 + x^2 + x^4 + x^6 + \dots)(1 + x^3 + x^6 + \dots)(1 + x^4 + x^8 + \dots),$$

and notice that each term is a geometric series! The product can be written as

$$\frac{1}{1-x} \cdot \frac{1}{1-x^2} \cdot \cdot \frac{1}{1-x^3} \dots$$

Thinking as a combinatorialist, we are not concerned about whether these series converge, since we consider the powers of x to be merely placeholders. These previous observations lead to Euler's Theorem.

Theorem 3 (Euler). *The generating function of the number of partitions of n is equal to*

$$\sum_{n=0}^{\infty} p(n)x^n = \prod_{k=1}^{\infty} \frac{1}{1-x^k}$$

Note that if we are interested in explicitly calculating some particular $p(n)$, we do not need the entire infinite product, or even any complete factor, since no partition of n can use any integer greater than n.

Example. Find p_8.

We expand

$$\begin{aligned}(1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + \dots)(1 + x^2 + x^4 + x^6 + x^8 \dots)(1 + x^3 + x^6 + \dots)\times \\ \times(1 + x^4 + x^8 + \dots)(1 + x^5 + \dots)(1 + x^6 + \dots)(1 + x^7 + \dots)(1 + x^8 + \dots) \\ = 1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + 11x^6 + 15x^7 + 22x^8 + \dots\end{aligned}$$

so $p(8) = 22$. Note that all of the coefficients prior to this are also correct, but the following coefficients are not necessarily the corresponding partition numbers.

Here is another way to do the computation: we define

$$f(x) = \prod_{k=1}^{8} \frac{1}{1 - x^k}$$

to make sure the coefficient will be correct. Instead of doing the explicit product above, we compute the Taylor series at $x = 0$ (maybe with the help of Matlab or Mathematica), which has the same effect.

References

- https://www.whitman.edu/mathematics/cgt_online/book/section03.03.html
- https://www2.math.upenn.edu/~wilf/PIMS/PIMSLectures.pdf