POINTWISE CONVERGENCE:

$\{f_n : M \to \mathbb{R}\}$ sequence in $F_b(M;\mathbb{R})$

f_n converges POINTWISE to f if $\boxed{\lim_{n\to\infty} f_n(x) = f(x)}$ $\forall x \in M$

$|f_n(x) - f(x)| \to 0$

UNIFORM CONVERGENCE:

$\{f_n : M \to \mathbb{R}\}$ seq. in $(F_b(M;\mathbb{R}), \|\cdot\|_\infty)$

f_n converges UNIFORMLY to f if $\boxed{\lim_{n\to\infty} \|f_n - f\|_\infty = 0}$

$\|f_n - f\|_\infty = \sup_{x\in M} |f_n(x) - f(x)|$

Remarks:

(i) uniform $\Rightarrow$ pointwise (uniform is STRONGER conv.)

$$\forall x \in M: \ |f_n(x) - f(x)| \le \sup_{y\in M} |f_n(y) - f(y)| = \|f_n - f\|_\infty \to 0$$

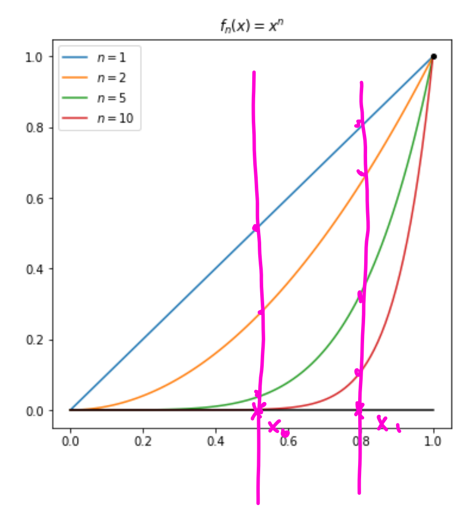
(ii) in general, pointwise $\not\Rightarrow$ uniform

e.g. $f_n(x) = x^n$, $x \in [0,1]$

$\forall x \in [0,1]$ (fixed!)

$$f_n(x) = x^n \longrightarrow \begin{cases} 0 & \text{if } 0 \le x < 1 \\ 1 & \text{if } x = 1 \end{cases}$$

(pointwise conv.)



<u>but</u> f_m does <u>NOT</u> converges uniformly to $f(x) = \begin{cases} 0 & 0 \le x < 1 \\ 1 & x = 1 \end{cases}$

assume it does, i.e. $\forall \varepsilon > 0 \ \exists N \in \mathbb{N}$ s.t. $\|f_m - f\|_\infty < \varepsilon$ $\forall m \geq N$

let $\varepsilon = \frac{1}{2}$: $\exists N \in \mathbb{N}$ s.t. $\underbrace{\|f_m - f\|_\infty}_{} < \frac{1}{2}$ $\forall m \geq N$, where $\sup_{x \in M} |f_m(x) - f(x)| = \|f_m - f\|_\infty$

$\Updownarrow$

$|f_m(x) - f(x)| < \frac{1}{2}$ $\forall m \geq N, \forall x \in [0,1]$

however for $m = N$ and $\tilde{x} = \left(\frac{3}{4}\right)^{\frac{1}{N}}$ we have

$$|f_N(\tilde{x}) - 0| = \left|\left(\left(\tfrac{3}{4}\right)^{\frac{1}{N}}\right)^N\right| = \frac{3}{4} > \frac{1}{2} \ \lightning$$

(the trick is that with <u>pointwise conv.</u> the <u>RATE</u> of convergence of $f_m(x)$ to $f(x)$ may depend on the point $x \in M$, while for <u>uniform conv.</u> the RATE is the same for all $x \in M$)