The closed unit ball in normed vector spaces

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MATH 3441 Real Analysis I

To complement our long journey into the theory metric spaces, we will see in these notes a necessary and sufficient condition for the closed unit ball to be a compact subset in a vector space.

Definition 1. Given a normed vector space $(X, \|\cdot\|)$, the closed unit ball is defined as

$$\overline{B(\mathbf{0}_X;1)} := \{x \in X \mid \|x\| \leq 1\} \subset X \ .$$

Theorem 2. *Given a normed vector space $(X, \|\cdot\|)$, the closed unit ball $\overline{B(\mathbf{0}_X;1)}$ is a compact subset of X if and only if X is finite dimensional.*

We will divide the proof in two parts, accounting for the case of finite and infinite dimensional vector spaces (on $\mathbb{R}$).

Proof. (part 1)
Suppose that X has finite dimension: $\dim X = k$. It is a fundamental result from Linear Algebra that there exists a linear isomorphism

$$\mathcal{T} : X \xrightarrow{\sim} \mathbb{R}^k$$
$$x = \sum_{j=1}^{k} \alpha_j e_j \mapsto (\alpha_1, \ldots, \alpha_k)$$

(it is actually an isometric isomorphism between $(X, \|\cdot\|_1)$ and $(\mathbb{R}^k, \|\cdot\|_1)$).
Since $\mathcal{T}$ is an isometry and $\overline{B(\mathbf{0}_X;1)}$ is bounded in X, then $\mathcal{T}(\overline{B(\mathbf{0}_X;1)})$ is bounded in $\mathbb{R}^k$. Furthermore, since $\mathcal{T}^{-1}$ is continuous and $\overline{B(\mathbf{0}_X;1)}$ is closed in X, $\mathcal{T}(\overline{B(\mathbf{0}_X;1)})$ is closed in $\mathbb{R}^k$.
From Heine-Borel Theorem, we can therefore conclude that $\mathcal{T}(\overline{B(\mathbf{0}_X;1)}) \subseteq \mathbb{R}^k$ is compact (closed and bounded) and, using continuity of $\mathcal{T}^{-1}$, it follows that $\overline{B(\mathbf{0}_X;1)}$ is compact in X. □

Before tackling the case of infinite dimensional vector spaces, we first state (and prove) the following lemma.

Lemma 3 (Riesz's Lemma). *Let $(X, \|\cdot\|)$ be a normed vector space and let $Z \subset X$ be a proper and closed linear subspace of X. Then there exists an element $x_0 \in X$ with $\|x_0\| = 1$ and such that $\|z - x_0\| \geq \frac{1}{2}$, for every $z \in Z$.*

Remark 4. *While here we chose the value $\frac{1}{2}$, Riesz's Lemma is valid for any value $\theta \in (0, 1)$.*

Proof. (Riesz's Lemma) Since Z is a closed, proper subset of X, there exists $\bar{x} \in X \setminus Z$ and there exists $r > 0$ such that $B(\bar{x}; r) \subset X \setminus Z$ (i.e. $X \setminus Z$ is open and $\bar{x}$ is an interior point). This implies that $\|\bar{x} - z\| \geq r$ for every $z \in Z$. Let d be the infimum of $\|z - \bar{x}\|$ over all elements in Z:

$$d := \inf_{z \in Z} \|\bar{x} - z\| \geq r > 0$$

By the definition of infimum, there exists $z^* \in Z$ such that

$$\|\bar{x} - z^*\| \leq 2d \; .$$

Consider now the following point in X:

$$x_0 := \frac{\bar{x} - z^*}{\|\bar{x} - z^*\|} \in X \; ;$$

by construction we have that $\|x_0\| = 1$. Furthermore, for any $z \in Z$ we have

$$\begin{aligned} x_0 - z &= \frac{\bar{x} - z^*}{\|\bar{x} - z^*\|} - z = \frac{1}{\|\bar{x} - z^*\|}\Big(\bar{x} - z^* - \|\bar{x} - z^*\|z\Big) \\ &= \frac{1}{\|\bar{x} - z^*\|}\Big(\bar{x} - \underbrace{(z^* + \|\bar{x} - z^*\|z)}_{\in Z}\Big) \end{aligned}$$

where we notice that, since Z is a linear subspace of X, $z^* + \|\bar{x} - z^*\|z$ is an element of Z.

Finally, $\forall\; z \in Z$

$$\|x_0 - z\| = \frac{\|\bar{x} - (z^* + \|\bar{x} - z^*\|z)\|}{\|\bar{x} - z^*\|} \geq \frac{d}{\|\bar{x} - z^*\|} \geq \frac{1}{2} \; .$$

□

We can finally proceed with concluding the proof of Theorem **??**.

Proof. (part 2) Let X be an infinite dimensional vector space. We will prove that $\overline{B(\mathbf{0}_X;1)}$ is *not* sequentially compact: the idea is to construct a sequence in $\overline{B(\mathbf{0}_X;1)}$ with no convergent subsequence.

Let $x_1 \in \overline{B(\mathbf{0}_X;1)}$ and call $X_1 := \langle x_1 \rangle$, i.e. X_1 is the 1-dimensional subspace of the linear combinations of x_1. Clearly, X_1 is a proper subset of X; additionally, X_1 is closed (because it has finite dimension). We can then apply Riesz's Lemma and fine a point $x_2 \in \overline{B(\mathbf{0}_X;1)}$ such that

$$\|x_2 - z\| \geq \frac{1}{2} \; , \qquad \forall\; z \in X_1 \; .$$

Consider now $X_2 := \langle x_1, x_2 \rangle$, i.e. X_2 is the 2-dimensional subspace of the linear combinations of x_1 and x_2. Similarly as for X_1, X_2 is a proper, closed linear subspace of X, therefore by Riesz's Lemma $\exists\; x_3 \in \overline{B(\mathbf{0}_X;1)}$ such that

$$\|x_3 - z\| \geq \frac{1}{2} \; , \qquad \forall\; z \in X_2 \; .$$

We proceed as above for any $n = 3, 4, \ldots$: from the previous $n-1$ steps, we consider the set $X_n := \langle x_1, \ldots, x_n \rangle$, where $x_j \in \overline{B(\mathbf{0}_X;1)}$ $\forall\; j = 1, \ldots, n$ and X_n is a proper, closed linear subspace of X ($\dim X_n = n$ vs $\dim X = \infty$). Therefore $\exists\; x_{n+1} \in \overline{B(\mathbf{0}_X;1)}$ such that

$$\|x_{n+1} - z\| \geq \frac{1}{2} \; , \qquad \forall\; z \in X_n \; ,$$

and so on.

We also notice that these subspaces are nested ($X_1 \subset X_2 \subset \ldots \subset X_n \subset X_{n+1} \subset \ldots$) and by construction

$$\|x_j - x_k\| \geq \frac{1}{2} \; , \qquad \forall\; j, k \in \mathbb{N} \; .$$

This implies that such a sequence $\{x_n\} \subset \overline{B(\mathbf{0}_X;1)}$ is *not* Cauchy. Thus, *any* subsequence $\{x_{n_k}\}$ is also *not* Cauchy, therefore it is not convergent. □